INTRODUCTION TO: CLASSIFICATION THEORY FOR ABSTRACT ELEMENTARY CLASS

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Content

Abstract

Classification theory of elementary classes deals with first order (elementary) classes of structures (i.e. fixing a set T of first order sentences, we investigate the class of models of T with the elementary submodel notion). It tries to find dividing lines, prove their consequences, prove "structure theorems, positive theorems" on those in the "low side" (in particular stable and superstable theories), and prove "non-structure, complexity theorems" on the "high side". It has started with categoricity and number of non-isomorphic models. It is probably recognized as the central part of model theory, however it will be even better to have such (non-trivial) theory for non-elementary classes. Note also that many classes of structures considered in algebra are not first order; some families of such classes are close to first order (say have kind of compactness). But here we shall deal with a classification theory for the more general case without assuming knowledge of the first order case (and in most parts not assuming knowledge of model theory at all).

§0 Introduction and notation

In §2 we shall try to explain the purpose of the book to mathematicians with little relevant background. §1 describes dividing lines and gives historical background. In §5 we point out the (reasonably limited) background needed for reading various parts and some basic definitions and in §6 we list the use of symbols. The content of the book is mostly described in §2-§3-§4 but §4 mainly deals with further problems and §6 with the symbols used.

Is this a book? I.e. is it a book or a collection of articles? Well, in content it is a book but the chapters have been written as articles, (in particular has independent introductions and there are some repetitions) and it was not clear that they will appear together, see §5(A) for more on how to read them.

§1 Introduction for model theorists

(A) <u>Why to be interested in dividing lines</u>?

Classification theory for first order (= elementary) classes is so established now that up to the last few years most people tended to forget that there are non-first order possibilities. There are several good reasons to consider these other possibilities; first, it is better to understand a more general context, we would like to prove stronger theorems by having wider context, classify a larger family of classes. Second, understanding more general contexts may shed light on the first order one. In particular, larger families may have stronger closure properties (see later). Third, many classes arising in "nature" are not first order ("in nature" here means other parts of mathematics).

Of course, we may suspect that applying to a wider context may leave us with little content, i.e., the proofs may essentially be just rewording of the old proofs (with cumbersome extra conditions); maybe there is no nice theory, not enough interesting things to be discovered in this context; it seems to me that experience has already refuted the first suspicion. Concerning the other suspicion, we shall try to give a positive answer to it, i.e. develop a theory; on both see the rest of the introduction.

In any case, "not first order" does not define our family of classes of models as discussed below. This is both witnessed from the history (on which this section concentrates) and suggested by reflection; clearly we cannot prove much on arbitrary classes, so we need some restriction to reasonable classes. Now there may be incomparable cases of reasonableness and a priori it is natural to expect to be able to say considerably more on the "more reasonable" cases. E.g. we expect that much more can be said on first order classes than on the class of models of a sentence from $\mathbb{L}_{\omega_1,\omega}$.

We are mainly interested here in generalizing the theorems on categoricity, superstability and stability to such contexts, in particular we consider the parallel of Łoś Conjecture and the (very probably much harder) main gap conjecture as test problems.

This choice of test problem is connected to the belief in (a),(b),(c) discussed below (that motivates [Sh:c]).

(a) It is very interesting to find dividing lines and it is a fruitful approach in investigating quite general classes of models.

That is, we start with a large family of (in our case) classes (e.g., the family of elementary (= first order) classes or the family of universal classes or the family of locally finite algebras satisfying some equations) and we would like to find natural

dividing lines. A dividing line is not just a good property, it is one for which we have some things to say on both sides: the classes having the property and the ones failing it. In our context normally all the classes on one side, the "high" one, will be provably "chaotic" by the non-structure side of our theory, and all the classes on the other side, the "low" one will have a positive theory. The class of models of true arithmetic is a prototypical example for a class in the "high" side and the class of algebraically closed field the prototypical non-trivial example in the "low" side.

Of course, not all important and interesting properties are like that. If F is a binary function on a set A, not much is known to follow from (A, F) not being a group. In model theory introducing o-minimal theories was motivated by looking for parallel to minimal theories and attempts to investigate theories close to the real field (e.g., adding the function $x \mapsto e^x$). Their investigation has been very important and successful, including parallels of stability theory for strongly minimal sets, but it does not follow our paradigm. A success of the guideline of looking for dividing lines had been the discovery of being stable (elementary classes, i.e. $(\mathrm{Mod}_T, \prec)$, [Sh 1]). From this point of view to discover a dividing line means to prove the existence of complementary properties from each side:

(i) T is unstable iff it has the order property (recall that T has the order property means that: some first order formula $\varphi(\bar{x}, \bar{y})$ linearly orders in M some infinite $\mathbf{I} \subseteq {}^{\ell g(\bar{x})}M$ in a model M of T)

(ii) T is stable iff $A \subseteq M \models T$ implies $|\mathbf{S}(A, M)|$, the set of 1-types on A for M is not too large $(\leq |A|^{|T|})$.

A case illustrating the point of dividing line is a precursor of the order property, property E of Ehrenfeucht [Eh57], it says that some first order formula $\varphi(x_1, \ldots, x_n)$ is asymmetric on some infinite $A \subseteq M, M$ a model of T; it is stronger than the order property (= negation of stability). A posteriori, order on the set of n-tuples is simpler; this is not a failure, what Ehrenfeucht did was fine for his aims, but looking for dividing lines forces you to get the "true" notion.

Even better than stable was superstable because it seems to me to maximize the "area" which we view as being how many elementary classes it covers times how much we can say about them. On the other hand, it has always seemed to me more interesting than $\aleph_0$-stable as the failure of $\aleph_0$-stability is weak, i.e. it has a few consequences. There is a first order superstable not $\aleph_0$-stable class K such that a model $M \in K$ is determined up to isomorphism by a dimension (a cardinal) and a set of reals. This exemplifies that an elementary class can fail to be $\aleph_0$-stable but still is "low": we largely can completely list its models. Such a class is the class of vector spaces over $\mathbb{Z}/2\mathbb{Z}$ expanded by predicates P_n for independent sub-spaces of co-dimension 2. A model M in this class is determined up to isomorphism by

one cardinal (the dimension of the sub-space $V_M = \cap\{P_n^M : n \in \mathbb{N}\}$) and the quotient M/V_M which has size at most continuum (alternatively the set $\{\eta_a : a \in M\}, \eta_a(n) \in \{0,1\}$ and where $\eta_a = \langle \eta_a(0), \eta_a(1), \ldots \rangle$ and $\eta_a(n) = 0 \Leftrightarrow a \in P_n^M$).

Of course, the guidelines of looking for dividing lines if taken religiously can lead you astray. It does not seem to recommend investigation of FMR (Finite Morley Rank) elementary classes which has covered important ground (see e.g. Borovik-Nessin [BoNe94]). This guideline has helped, e.g. to discover dependent and strongly dependent elementary classes, but so far our approach has seemingly not succeeded too much in advancing the investigation.

See more on this in end of §2(B), in particular Question 2.15.

(b) It is desirable to have an exterior a priori existing goal as a test problem.

Such a problem in model theory was Łos conjecture which says: if a first order class of countable vocabulary (= language) is categorical in one $\lambda > \aleph_0$ (= has one and only one model of cardinality λ up to isomorphisms) then it is categorical in every $\lambda > \aleph_0$. At least for me so was Morley conjecture [Mo65] which says that for first order class with countable vocabulary, the number of its models of cardinality $\lambda > \aleph_0$ up to isomorphism is non-decreasing with λ. This motivated my research in the early seventies which eventually appeared as [Sh:a] (with several late additions like local weight in [Sh:a, Ch.V,§4]). Now having introduced "$\aleph_\varepsilon$-saturated models", it seems unconvincing to understand $\dot{I}(\lambda, K)$, the number of models in K of cardinality λ up to isomorphism, for K the class of $\aleph_\varepsilon$-saturated models of a first order class, hence though essentially done then, was not written till much later. Eventually "$\dot{I}(\lambda, T)$ non-decreasing" was done for the family of classes of models of a countable first order theory (which was the original center of interest; see [Sh:c]).

By this solution, there are very few "reasons" for such $K = \text{Mod}_T$ to have many models: being unstable, unsuperstable, DOP (dimensional order property), OTOP (omitting type order property) and deepness (for fuller explanation see after 2.12; see more, characterizing the family of functions $\dot{I}(\lambda, T)$ for countable T in Hart-Hrushovski-Laskowski [HHL00]). So the direct aim was to solve the test question (e.g., the main gap[1]), but the motivation has always been the belief that solving it will be rewarded with discovering worthwhile dividing lines and developing a theory for both sides of each.

The point is that looking at the number of non-isomorphic models and in particular the main gap we hope to develop a theory. Other exterior problems will hopefully give rise to other interesting theories, which may be related to stability

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[1]which says that <u>either</u> $\dot{I}(\lambda, T) = 2^\lambda$ for every ($> |T|$, or large enough) λ <u>or</u> $\dot{I}(\aleph_\alpha, T) \leq \beth_{\gamma(T)}(|\alpha|)$ for every α (for some ordinal $\gamma(T)$); see more in 2.10.

theory or may not; this was the point of [Sh 10], in particular the long list of exterior results in the end of its introduction, and the words "classification theory" in the name of [Sh:a]. But, the above point seemingly was slow in being noticed.

Of course, if we consider the family of classes which are "high" by one criterion/dividing line, we expect that with respect to other questions/dividing lines the "previously high ones" will be divided and on a significant portion of them we have another positive theory, quite reasonably generalizing the older ones (but maybe we shall be led to very different theories). E.g. for unstable first order classes [Sh:93] succeeded in this respect: "low ones" are the simple theories and the "high ones" are theories with the tree property (on exciting later developments, see [KiPi98] or [GIL02]).

   (c) successful dividing lines will throw light on problems not considered when suggesting them.

The point is that the theory should be worthwhile even if you discard the original test problems. Stability theory is just as interesting for some other problems as for counting number of non-isomorphic models. E.g.

   $(*)_1$ the maximal number of models no one embeddable into another.

This sounds very close to counting, so we expect this is to have a closely related answer.

In fact for elementary classes (with countable vocabulary) which have a structure theorem (see 2.10 below), this number is $< \beth_{\omega_1}$, for the others it is very much higher (see more on the trichotomy after 2.12); so the answer to $(*)_1$ turns out to be nicer than the one concerning the number, $\lambda \mapsto \dot{I}(\lambda, T)$.

   $(*)_2$ in $\mathfrak{K}$ there are models very similar yet non-isomorphic.

This admits several interpretations which in general have complete and partial solutions quite tied up with stability theory. One is finding $\mathbb{L}_{\infty,\lambda}$-equivalent not isomorphic models of cardinality λ. Stronger along this line are EF_λ-equivalent not isomorphic. Another is that there are non-isomorphic models of T such that a forcing neither collapsing cardinals nor adding too short sequences makes them isomorphic. For non-logicians we should explain that this says in a very strong sense that there are no reasonable invariants, see [Sh 225], [Sh 225a], Baldwin-Shelah [BLSh 464], Laskowski-Shelah [LwSh 489], Hyttinen-Tuuri [HyTu91], Hyttinen-Shelah-Tuuri [HShT 428], Hyttinen-Shelah [HySh 474], [HySh 529], [HySh 602].

   $(*)_3$ For which classes K do we have: its models are no more complicated than trees (in the graph theoretic sense say rooted graphs with no cycle)?

This question was specified to having a tree of submodels which is "free" (= "non-forking") and it is a decomposition, i.e., the whole model is prime over the tree. This is answered by stability theory (for Mod_T, T countable)

(∗)$_4$ similarly replacing graphs with no cycles by another simple class, e.g., linear orders.

This is very interesting, but too hard at present (see more in Cohen-Shelah [CoSh:919])

(∗)$_5$ decidable theories, e.g. we may note that there was much done on decidability and understanding of the monadic theory of some structures (in particular Rabin's celebrated theorem). Those works concentrated on linear orders and on trees. Was this because of our shortcoming or for inherent reasons?

We may interpret this as a call to classify classes, in particular, first order ones by their complexity as measured by monadic logic. This was carried to large extent in Baldwin-Shelah [BlSh 156] for first order classes. Now this seems a priori orthogonal to classification taking number of models as the test question; note that the class of linear orders is unstable but reasonably low for [BlSh 156], whereas any class is maximally complicated if it has a pairing function (e.g. a one-to-one function F^M from $P_1^M \times P_2^M$ into P_3^M while P_1^M, P_2^M are infinite) and there are such classes which are categorical in every $\lambda \geq \aleph_0$. In spite of all this [BlSh 156] relies heavily on stability theory; see [Bl85], [Sh 197], [Sh 205], [Sh 284c]

(∗)$_6$ the ordinal κ-depth of a model (Karp complexity).

For a model M and a partial automorphism f of M, $\mathrm{Dom}(f)$ of cardinality $< \kappa$, we can define its κ-depth in M, an ordinal (or ∞) by $\mathrm{Dp}_\kappa(f, M) \geq \alpha$ iff for every $\beta < \alpha$ and subsets A_1, A_2 of cardinality $< \kappa$, there is a partial automorphism f' of M extending f of κ-depth $\geq \beta$ such that $|\mathrm{Dom}(f')| < \kappa, A_1 \subseteq \mathrm{Dom}(f'), A_2 \subseteq \mathrm{Rang}(f')$.
Let

$$\mathrm{Dp}_\kappa(M) = \cup\{\mathrm{Dp}_\kappa(f, M) + 1 : f \text{ a partial automorphism of } M \text{ of cardinality}$$
$$< \kappa \text{ and } \mathrm{Dp}_M(f) < \infty\}.$$

This measures the complexity of the models and $\mathrm{Dp}_\kappa(T) = \cup\{\mathrm{Dp}(M) + 1 : M$ a model of $T\}$ is a reasonable measure of the complexity of T. With considerable efforts, reasonable knowledge concerning this measure was gained by Laskowski-Shelah [LwSh 560], [LwSh 687], [LwSh 871] confirming to some extent the thesis above.

$(*)_7$ categoricity and number of models in $\aleph_\alpha$, in ZF (i.e., with no choice).

See [Sh 840].

You may view in this context the question of having non-forking (= abstracts dependence relations), orthogonality, regularity but for me this is part of the inside theory rather than an external problem

(d)   non-structure is not so negative.

Now this book predominantly deals with the positive side, structure theory, so defending the honour of non-structure is not really necessary (it is the subject of [Sh:e] though). Still first we may note that finding the maximal family of classes for which we know something is considerably better than finding a sufficient condition. In particular finding "the maximal family ... such that ..." is finding dividing lines and this is meaningless without non-structure results.

Second, this forces you to encounter real difficulties and develop better tools; also using the complicated properties of a class which already satisfies some "low side properties" may require using and/or developing a positive theory.

Last but not least, non-structure from a different perspective is positive. Applying "non-structure theory" to modules this gives representation theorems of rings as endomorphism rings (see Göbel-Trlifaj [GbTl06]; note that the "black boxes" used there started from [Sh:c, VIII]). In fact, generally for unstable elementary class K, we can find models which in some respect represent a pregiven ordered group (see [Sh 800]). This has been applied to clarify in some cases to which generalized quantifiers give a compact logic (see [Sh:e] and more in [Sh 800]).

It may clarify to consider an alternative strategy: we have a reasonable idea of what we look for and we have a specific class or structure which should fit the theory. This works when the analysis we have in mind is reflected reasonably well in the specific case. It may be misleading when the examples we have, do not reflect the complexity of the situation, and it seems to be the case in the problems we have at hand. More specifically, though the "example" of the theory of superstable first order classes stand before us, we do not try to take the way of trying to assume enough of its properties so that it works; rather we try look for dividing lines.
See more on "why dividing lines" in the end of (B) of §2.

(B) <u>Historical comments on non-elementary classes</u>:

Let us return to non-elementary classes. Generally, on model theory for non-elementary classes see Keisler [Ke71] and the handbook [BaFe85]: closer to our interest is the forthcoming book of Baldwin [Bal0x] and the older Makowsky [Mw85], mainly around $\aleph_1$.

Below we present the results according to the kind of classes dealt with (rather than chronologically).

The oldest choice of families of classes (in this context) is the family of class of κ-sequence homogeneous models for a fixed D.

Morley and Keisler [KM67] proved that there are at most $2^{2^{|T|}}$ such models of T in any cardinality. Keisler [Ke71] proved that if $\psi \in \mathbb{L}_{\omega_1,\omega}$ is categorical in $\aleph_1$ and its model in $\aleph_1$ is sequence homogeneous then it is categorical in every $\lambda > \aleph_1$; generalizing (his version of) the proof of Morley's theorem. In [Sh 3] instead of having a monster $\mathfrak{C}$, i.e., a $\bar{\kappa}$-saturated model of a first order T, we have a $\bar{\kappa}$-sequence homogeneous model $\mathfrak{C}$. Let $D = D(\mathfrak{C}) = \{\mathrm{tp}(\bar{a}, \emptyset, \mathfrak{C}) : \bar{a} \in \mathfrak{C}$; i.e., $\bar{a}$ a finite sequence from $\mathfrak{C}\}$; note that $D, \bar{\kappa}$ determines $\mathfrak{C}$ and we look at the class of $M \prec \mathfrak{C}$ (or the class of (D, λ)-homogeneous $M \prec \mathfrak{C}$). There the stability spectrum was reasonably characterized, splitting and strong splitting were introduced (for first order theory this was later refined to forking). See somewhat more in [Sh 54].

Lately, this (looking at the $\prec$-submodels of a (D, λ)-homogeneous monster $\mathfrak{C}$) has become very popular, see Hyttinen [Hy98], Hyttinen and Shelah [HySh 629], [HySh 632], [HySh 629] (the main gap for $(D, \aleph_\varepsilon)$-homogeneous models for a good diagram D), Grossberg-Lessman [GrLe02], [GrLe0x] (the main gap for good $\aleph_0$-stable (= totally transcendental)), [GrLe00a], Lessman [Le0x], [Le0y] (all on generalizing geometric stability).

We may look at contexts which are closer to first order, i.e., having some version of compactness. Chang-Keisler [ChKe62], [ChKe66] has looked at models with truth values in a topological space such that ultraproducts can be naturally defined. Robinson had looked at model theory of the classes of existentially closed models of first order universal or just inductive theories. Henson [He74] and Stern [Str76] have looked at Banach spaces (we can take an ultraproduct of the spaces, throw away the elements with infinite norm and divide by those with infinitesimal norm). Basically the logic is "negation deficient", see Henson-Iovino [HeIo02].

The aim of [Sh 54] was to show that the most basic stability theory was doable for Robinson style model theory. In particular it deals with case II (the models of a universal first order theory which has the amalgamation property) and case III (the existentially closed models of a first order inductive ($= \Pi_2^1$) theory); those are particular cases of (D, λ)-homogeneous models. Case II is a special case of III where T has amalgamation. Lately, Hrushovski dealt with Robinson classes (= case II above). A Ph.D. student of mine in the seventies was supposed to deal with Banach spaces but this has not materialized. Henson and Iovino continued to develop model theory of Banach spaces. Lately, interest in the classification theory in such contexts has awakened and dealing with cases II and III and complete metric spaces and Banach spaces and relatives, now called continuous model theory, see Ben-Yaacov [BY0y], Ben-Yaacov Usvyatsov [BeUs0x], Pillay [Pi0x], Shelah-Usvyatson [ShUs 837].

The most natural stronger (than first order) logic to try to look at, in this

context, has been $\mathbb{L}_{\omega_1,\omega}$ and even $\mathbb{L}_{\lambda^+,\omega}$. By 1970 much was known on $\mathbb{L}_{\omega_1,\omega}$ (see Keisler's book [Ke71]); however, if you do not like non-first order logics, look at the class of atomic models of a countable first order T. The general question looks hard. At the early seventies I have clarified some things on $\psi \in \mathbb{L}_{\omega_1,\omega}$ categorical in $\aleph_1$, but it was not clear whether this leads to anything interesting. Then the following question of Baldwin catches my eye (question 21 of the Friedman list [Fr75])

$(*)_1$ can $\psi \in \mathbb{L}(\mathbf{Q})$ have exactly one uncountable model up to isomorphism?
$\mathbf{Q}$ stands for the quantifier "there are uncountably many"

This is an excellent question, a partial answer was ([Sh 48])

$(*)_2$ if $\Diamond_{\aleph_1}$ and $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ has at least one but $< 2^{\aleph_1}$ models in $\aleph_1$ up to isomorphism <u>then</u> it has a model in $\aleph_2$ (hence has at least 2 non-isomorphic models)

Only later the original problem (even for $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$) was solved in ZFC, see below. It seems natural to ask in this case how many models ψ has in $\aleph_2$, and then successively in $\aleph_n$ (raised in [Sh 48]), but as it was hard enough, the work concentrates on the case of $\psi \in \mathbb{L}_{\omega_1,\omega}$, so ([Sh 87a], [Sh 87b] and generalizing it to cardinals $\lambda, \lambda^+, ...$ is a major aim of this book):

$(*)_3$ (a) if $n < \omega, 2^{\aleph_0} < 2^{\aleph_1} < \ldots < 2^{\aleph_n}, \psi \in \mathbb{L}_{\omega_1,\omega}, \dot{I}(\aleph_\ell, \psi) < \mu_{\mathrm{wd}}(\aleph_\ell)$, for[2] $\ell \le n$ and $\dot{I}(\aleph_1, \psi) \ge 1$ <u>then</u> ψ has a model in $\aleph_{n+1}$ and without loss of generality ψ is categorical in $\aleph_0$

(b) if the assumption of (a) holds for every $n < \omega$ and ψ is for simplicity categorical in $\aleph_0$ <u>then</u> the class Mod_ψ is so-called excellent (see (c))

(c) if $\psi \in \mathbb{L}_{\omega_1,\omega}$ is excellent and is categorical in one $\lambda > \aleph_0$ then it is categorical in every $\lambda > \aleph_0$.

Essentially, it was proved that excellent $\psi \in \mathbb{L}_{\omega_1,\omega}$ are very similar to $\aleph_0$-stable (= totally transcendental) first order countable theories (after some "doctoring"). The set of types over a model $M, \mathscr{S}(M)$ is restricted (to not violate the omission of the types which every model of ψ omit). The types themselves are as in the first order case, set of formulas but we should not look at complete types over any $A \subseteq M \models \psi$, only at the cases $A = N \prec M$ or $A = M_1 \cup M_2$ where M_1, M_2 are stably amalgamated over M_0 and more generally at $\cup\{M_u : u \in \mathscr{P}^-(n)\}$, where $\langle M_u : u \in \mathscr{P}^-(n)\rangle$ is a "stable system".

This work was continued in Grossberg and Hart [GrHa89], (main gap), Mekler and Shelah [MkSh 366] (dealing with free algebras), Hart and Shelah [HaSh 323]

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[2] $\mu_{\mathrm{wd}}(\aleph_\ell)$ is "almost" equal to $2^{\aleph_\ell}$

(categoricity may hold for $\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_n$ but fail for large enough λ) and lately Zilber [Zi0xa], [Zi0xb] (connected to his programs). Further works on more general but not fully general are [Sh 300], Chapter II (universal classes), Shelah and Villaveces [ShVi 635], van Dieren [Va02] (abstract elementary class with no maximal models). See also the closely related Grossberg and Shelah [GrSh 222], [GrSh 238], [GrSh 259], [Sh 394], (abstract elementary class with amalgamation), Grossberg [Gr91] and Baldwin and Shelah [BlSh 330], [BlSh 360], [BlSh 393]. Lately, Grossberg and VanDieren [GrVa0xa], [GrVa0xb] Baldwin-Kueker-VanDieren [BKV0x] investigate the related tame abstract elementary class including upward categoricity. They prove independently of IV.? that tame a.e.c. with amalgamation has nice categoricity spectrum; i.e. prove categoricity in cardinals $> \mu$ in the relevant cases; in the notation here "tame" means locality of orbital types over saturated model; on IV.?, see §4(B) after $(**)_\lambda$. Concerning $\mathbb{L}_{\kappa,\omega}$, see Makkai-Shelah [MaSh 285] (on cateogoricity of $T \subseteq \mathbb{L}_{\kappa,\omega}, \kappa$ compact starting with λ successor), Kolman-Shelah [KlSh 362] ($T \subseteq \mathbb{L}_{\kappa,\omega}, \kappa$ measurable, amalgamation derived from categoricity), [Sh 472] ($T \subseteq \mathbb{L}_{\kappa,\omega}, \kappa$ measurable, only down from successor). See more in the book [Bal0x] of Baldwin on the subject.

Going back, $(*)_3$ deals with $\psi \in \mathbb{L}_{\omega_1,\omega}$, it generalizes the case $n = 1$ which, however, deals with $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$. On the other hand, $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ is not a persuasive end of the story as there are similar stronger logics. Also the proof deals with $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ in an indirect way, we look at a related class K which has also countable models but some first order definable set should not change when extending. So it seems that the basic notion is the right version of elementary extensions. This leads to analysis which suggests the notion of abstract elementary class, $\mathfrak{K}$ with $\mathrm{LST}(\mathfrak{K}) \le \aleph_0$ which, moreover, is $\mathrm{PC}_{\aleph_0}$ (in [Sh 88], represented here in Chapter I).

Now much earlier Jonsson [Jn56], [Jn60] had considered axiomatizing classes of models. Compared with the abstract elementary classes used (much later) in [Sh 88]=Chapter I, the main[3] differences are that he uses the order $\subseteq$ (being a submodel) on K (rather than an abstract order $\le_\mathfrak{K}$) and assume the amalgamation

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[3]Jonsson axioms were, in our notations, (for a fix vocabulary τ, finite in [Jn56], countable in [Jn60]), K is a class of τ-models satisfying

(I) there are non-isomorphic $M, N \in K$ in [Jn56]

$(I)'$ K has members of arbitrarily large cardinality in [Jn60]

(II) K is closed under isomorphisms

(III) the joint embedding property

(IV) disjoint amalgamation in [Jn56]

$(IV)'$ amalgamation in [Jn60]

(V) $\cup\{M_\alpha : \alpha < \delta\} \in K$ if $M_\alpha \in K$ is $\subseteq$-increasing

(and JEP joint embedding property). His aim was to construct and axiomatize
the construction of universal and then universal homogeneous models so includ-
ing amalgamation was natural; Morley-Vaught [MoVa62] use this for elementary
class. In fact if we add amalgamation (and JEP) to abstract elementary classes
we get such theorems (see I§2, in fact we also get uniqueness in a case of some-
what different character, I.?). From our perspective amalgamation (also $\leq_{\mathfrak{K}}=\subseteq$) is
a heavy assumption (but an important property, see later). Now, model theorists
have preferred saturated to universal homogeneous and prefer first order classes
(Morley-Vaught [MoVa62], Keisler replete) with very good reasons, as it is better
(more transparent and give more) to deal with one element than a model. That
is, assume our aim is to show that N from our class K is universal, i.e., we are
given $M \in K$ of cardinality not larger than that of N and we have to construct
an (appropriate) embedding of M into N. Naturally, we do it by approximations
of cardinality smaller than $\|M\|$, the number of elements of M. Jonsson uses as
approximations isomorphisms f from a submodel M' of M of cardinality $< \|M\|$.
Morley and Vaught use functions from a subset A of M into N such that: if
$n < \omega, a_0, \ldots, a_{n-1} \in A$ satisfy a first order formula in M then their image satisfies
it in N. So they have to add one element at each step which is better than dealing
with a structure. In fact, also in this book, for a different notion of type, the types
of elements continue to play a major role (but we use types which are not sets of
formulas over models). So we try to have "the best of both approaches" - all is done
over models from K, but we ask existence, etc., only of singletons, for this reason
in the proof of the uniqueness of "saturated" models we have to go "outside" the
two models, build a third (see V.B.? or II.?).

Here we have chosen abstract elementary class as the main direction. This
includes classes defined by $\psi \in \mathbb{L}_{\omega_1, \omega}$ and we can analyze models of $\psi \in \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$
in such context by a reduction. In [Sh 88] = Chapter I Baldwin's question was
solved in ZFC. Also superlimit models were introduced and amalgamation in λ was
proved assuming categoricity in λ and $1 \leq \dot{I}(\lambda^+, \mathfrak{K}) < 2^{\lambda^+}$ when $2^\lambda < 2^{\lambda^+}$. The
intention of the work was to prepare the ground for generalizing [Sh 87b]. Note
that sections §4,§5 from Chapter I are harder than the parallel in [Sh 87a] because
we deal with abstract elementary class (not just $\psi \in \mathbb{L}_{\omega_1, \omega}(\mathbf{Q})$).

Now [Sh 300] deals with universal classes. This family is incomparable with first

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(VI) if $N \in K$ and $M \subseteq N$ (so $|M| \neq \emptyset$ but not necessarily $M \in K$) and $\alpha > 0, \|M\| < \aleph_\alpha$
then there is $M' \in K$ such that $M \subseteq M' \subseteq N$ and $\|M'\| < \aleph_\alpha$ (this is a strong form of
the LST property).

Note that for an abstract elementary class $(K, \leq_{\mathfrak{K}})$, if $\leq_{\mathfrak{K}}=\subseteq \upharpoonright K$, then AxIV (smoothness) and
AxV (if $M_1 \subseteq M_2$ are $\leq_{\mathfrak{K}}$-submodels of N then $M_1 \leq_{\mathfrak{K}} M_2$) of I.? or II.? and part of AxI
become trivial (hence are missing from Jonsson axioms), the others give II, and a weaker form of
VI (specifically, for one $\aleph_\alpha$, i.e. $\aleph_\alpha = \mathrm{LST}(\mathfrak{K})^+$, the other cases are proved).

order and [Sh 155] gives hope it will be easier. Note that in excellent classes the types are set of formulas and this is true even for Chapter I though the so-called materializing replaces realizing a type. In [Sh 300] (orbital)-type is defined by $\leq_{\mathfrak{K}}$-mapping. Surprisingly we can still show "λ-universal homogeneous" is equivalent to λ-saturated under the reasonable interpretations (so have to find an element rather than a copy of a model) what was a strong argument for sequence homogenous models (rather than model homogeneous).

In [Sh 576], which is a prequel of the work here, (redone in [Sh:E46]) we generalize [Sh 88] to any abstract elementary class $\mathfrak{K}$ having no remnant of compactness, see on it below. On Chapter II, Chapter III see later.

I thank the institutions in which various parts of this book were presented and the student and non-students who heard and commented. Earlier versions of [Sh 300a], [Sh 300b], [Sh:e, III], [Sh 300c], [Sh 300d], [Sh 300e] were presented in Rutgers in 1986; some other parts were represented some other time. In Helsinki 1990 a lecture was on the indiscernibility from [Sh 300f], [Sh 300g]. First version of [Sh 576] was presented in seminars in the Hebrew University, Fall '94. The Gödel lecture in Madison Spring 1996 was on [Sh 576] and Chapter II. The author's lecture in the logic methodology and history of science, Kracow '99, was on Chapter II and Chapter III. In seminars at the Hebrew University, Chapter I was presented in Spring 2002, [Sh 576] was presented in 98/99, Chapter II + Chapter IV were presented in 99/00, Chapter II + Chapter III were presented in 01/02 and my lecture in the Helsinki 2003 ASL meeting was on good λ-frames and Chapter IV.

I thank John Baldwin, Emanuel Dror-Farajun, Wilfred Hodges, Gil Kalai, Adi Jarden, Alon Siton, Alex Usvyatsov, Andres Villaveces for many helpful comments and error detecting in the introduction (i.e. Chapter N).

Last, but not least, I thank Alice Leonhardt for beautifully typesetting the contents of this book.

§2 Introduction for the logically challenged

(This is recommended reading for logicians too, but there are some repetitions of part (A) of §1).

This is mainly an introduction to Chapter II, Chapter III.

> We assume the reader knows the notion of an infinite cardinal <u>but not</u> that he knows about first order logic (and first order theories); for reading (most of) the book, not much more is needed, see §5.
>
> Paragraphs assuming more knowledge or are not so essential will be in indented, e.g. when a result is explained ignoring some qualifications and we comment on them in indented text.

(A) <u>What are we after?</u>

This introduction is intended for a general mathematical audience. We may view our aim in this book as developing a theory dealing with abstract classes of mathematical structures that will also be referred to as models. Examples of structures are the field $\mathbb{R}$, any group and any ring. The classes of models we consider are called "abstract elementary classes" or briefly a.e.c. An abstract elementary class $\mathfrak{K}$ is a class of structures denoted by K together with an order relation denoted by $\leq_{\mathfrak{K}}$ which distinguishes for each structure N a certain family $\{M \in K : M \leq_{\mathfrak{K}} N\}$ of substructures (= submodels).

First, rather than giving a formal definition, we will give several examples:

<u>2.1 Examples</u>:

(i) the class of groups where the order relation is "being a subgroup".

In this example $\leq_{\mathfrak{K}}$ is simply being substructures. (In the sequel when we do not specify the order relation is means simply to take all substructures).

(ii) The class of algebraically closed fields with characteristic zero
(iii) the class of rings
(iv) the class of nil rings, i.e. ring R such that for every $x \in R$, $x^n = 0$ for some $n \geq 1$
(v) the class of torsion R-modules for a fixed ring R
(vi) the class of R-modules for a fixed ring R but unlike the previous cases the relation of $\leq_{\mathfrak{K}}$ is not just being a submodule, but being a "pure submodule"[4]

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[4]A left R-module M is a pure submodule of a left R-module N when if $rx = y, x \in N$ and $y \in M$ then $rx' = y$ for some $x' \in M$

(*vii*) the class of rings but $R_1 \leq_{\mathfrak{K}} R_2$ means here: R_1 is a subring of R_2 and if R_2' is a finitely generated subring of R_2 then $R_1 \cap R_2'$ is a finitely generated subring of R_1

(*viii*) the class of partial orders

(*ix*) concerning Hill Lemma, Baldwin, Eklof and Trlifaj [BETp06] show it fit in a.e.c. context.

Abstract elementary class form an extension of the notion of elementary class which mean a class of structures which are models of a so-called first order theory. The notion of abstract elementary classes, while more general, does not rely on elementary classes and indeed, for reading this introduction we do not assume knowledge of first order logic.

We will be mainly interested in this book in finding parallel to the "superstability theory" which is part of the "classification theory" (this is explained below; on the first order case see, e.g. [Sh:c], [Sh 200] or other books on the subject, e.g. Baldwin [Bal88]).

Superstability theory can be described as dealing with elementary classes of structures for which there is a good dimension theory; see on our broader aim below.

A structure M will have a so-called vocabulary τ_M (this is its "kind", e.g. is it a ring or a group). Note that for each class $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ we shall consider, all $M \in K$ has the same vocabulary (sometimes called language), which we denote by $\tau = \tau_{\mathfrak{K}}$, e.g., for a class of fields it is $\{+, \times, 0, 1\}$ where $+, \times$ are binary functions symbols interpreted in each field as two-place functions and similarly $0, 1$ are individual constant symbols. We may have also relations, (in example (*viii*) the partial order is a relation), note that relation symbols are usually called predicates. The reader may restrict himself to the case of countable or even finite vocabulary with function symbols only. We certainly demand each function symbol to have finitely many places (and similarly for relation symbols).

We try now, probably prematurely, to give exact definitions of some basic notions toward what long term goal we would like to advance, probably it will make more sense after/if the reader continues to read the introduction. (But most of this will be repeated and expanded).

We think that the family of abstract elementary classes $\mathfrak{K}$ (defined in 2.2 below) can be divided, in some ways, so that we can say significant things both on the "low", simple side and on the "high, complicated" side. This sounds vague, can we already state a conjecture? It seems reasonable that a class K with a unique member (up to isomorphism, of course) in a cardinality λ is simple; but what can be the class of cardinals for which this holds? This class is called the "categoricity spectrum of the abstract elementary class $\mathfrak{K}$" (see Definitions 2.2, 2.3 below), we conjecture that is a simple set, e.g. contains every large enough cardinal <u>or</u> does

not contain every large enough cardinal. Moreover, this also applies to the so-called superlimit spectrum of $\mathfrak{K}$ (see Definition 2.4). In the "low, simple" case we have, e.g. a dimension theory for $\mathfrak{K}$, and in the "high case" we can prove the class is complicated and so cannot have such a nice theory (this paragraph will be explained/expanded later).

Here we make some advances in this direction.
First, what exactly is an abstract elementary class? It is much easier to explain than the so-called "elementary classes" which is defined using (first order) logic. A major feature are closure under isormorphism and unions.

2.2 Definition. $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ is an abstract elementary class <u>when</u>

(A) (a) K is a class of structures all of the same "kind", i.e. vocabulary;
e.g. they can be all rings or all graphs, τ denote a vocabulary

    (b) K is closed under isomorphisms

    (c) $\leq_{\mathfrak{K}}$ is a partial order of K, also closed under isomorphisms
and $M \leq_{\mathfrak{K}} N$ implies $M \subseteq N, M$ a substructure of N and, of
course, $M \in K \Rightarrow M \leq_{\mathfrak{K}} M$

    (d) K (and $\leq_{\mathfrak{K}}$) are closed under direct limits, or, what is equivalent,
by unions of $\leq_{\mathfrak{K}}$-increasing chains, i.e. if I is a linear order and
$M_t(t \in I)$ is $\leq_{\mathfrak{K}}$-increasing with t then $M = \cup\{M_t : t \in I\}$ belongs
to K and; morever, $t \in I \Rightarrow M_t \leq_{\mathfrak{K}} M$

    (e) similarly to clause (d) inside $N \in K$, i.e., if $t \in I \Rightarrow M_t \leq_{\mathfrak{K}} N$
then $M \leq_{\mathfrak{K}} N$.

Two further demands are only slightly heavier

(B) (f) if[5] $M_\ell \leq_{\mathfrak{K}} N$ for $\ell = 1, 2$ and $M_1 \subseteq M_2$ then $M_1 \leq_{\mathfrak{K}} M_2$

    (g) $(K, \leq_{\mathfrak{K}})$ has countable character, which means that every structure
can be approximated by countable ones; i.e., if $N \in K$ then
every countable set of elements of N is included in some
countable $M \leq_{\mathfrak{K}} N$ (in the book but not in the
introduction we allow replacing "countable" by "of cardinality
$\leq$ LST$(\mathfrak{K})$" for some fixed cardinality LST$(\mathfrak{K})$).

Not all natural classes are included, e.g. the class of Banach spaces is not, as completeness is not preserved by unions of increasing chains. Still it seems very broad and the question is can we prove something in such a general setting.

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[5]this certainly holds if $\leq_{\mathfrak{K}}$ is defined as $\prec_{\mathscr{L}(\tau(\mathfrak{K}))}$ for some logic $\mathscr{L}$

2.3 Definition. 1) $\mathfrak{K}$ (or K) is categorical in λ <u>when</u> it has one and only one model of cardinality λ up to isomorphism.
2) The categoricity spectrum of $\mathfrak{K}$, cat($\mathfrak{K}$), is the class of cardinals λ in which $\mathfrak{K}$ is categorical.

A central notion in model theory is elementary classes or first order classes which are defined using so called first order logic (which the general reader is not required here to know, it is explained in the indented text below).

Each such class is the class of models of a first order theory with the partial order $\prec$.

Among elementary classes, a major division is between the so-called superstable ones and the non-superstable ones, and for each superstable one there is a dimension theory (in the sense of the dimension of a vector space). Our long term aim in restricted terms is to find such good divisions for abstract elementary classes, though we do not like to dwell on this further now, it seems user-unfriendly not to define them at all, so for the time being noting that for elementary classes being superstable is equivalent to having a superlimit model in every large enough cardinality; also noting that superstability for abstract elementary classes suffer from schizophrenia, i.e. there are several different definitions which are equivalent for elementary classes, the one below is one of them.

2.4 Definition. Let $\mathfrak{K}$ be an abstract elementary class.
1) We say f is a $\leq_{\mathfrak{K}}$-embedding of M into N when f is an isomorphism of M onto some $M' \leq_{\mathfrak{K}} N$.
2) $\mathfrak{K}_\lambda = (K_\lambda, \leq_{\mathfrak{K}_\lambda})$ where $K_\lambda = \{M \in \mathfrak{K} : \|M\| = \lambda\}$ and $\leq_{\mathfrak{K}_\lambda} = \leq_{\mathfrak{K}} \restriction K_\lambda$.
3) An abstract elementary class $\mathfrak{K}$ is superstable <u>iff</u> for every large enough λ, there is a superlimit structure M for $\mathfrak{K}$ of cardinality λ; where
4) We say that M is a superlimit (for $\mathfrak{K}$) when for some (unique) λ

   (a) $M \in \mathfrak{K}$ has cardinality λ

   (b) M is $\leq_{\mathfrak{K}}$-universal, i.e., if $M' \in K_\lambda$ then there is a $\leq_{\mathfrak{K}}$-embedding of M' into M, in fact with range $\neq M$

   (c) for any $\leq_{\mathfrak{K}}$-increasing chain of models isomorphic to M with union of cardinality λ, the union is isomorphic to M.

5) The superlimit spectrum of $\mathfrak{K}$ is the class of λ such that there is a superlimit model for $\mathfrak{K}$ of cardinality λ.

We shall return to those notions later.

What about the examples listed above? Concerning the strict definition of elementary classes as classes of the form $(\text{Mod}_T, \prec)$ defined below, among the examples in 2.1 the class of algebraically closed fields (example (ii)) is an elementary class

since it can be proved that being a sub-field is equivalent to being an elementary substructure for such fields.

In the example (i), the class of models is elementary, i.e., equal to Mod_T: the class of groups, but the order is not $\prec$ but $\subseteq$. This is true also in the examples (iii), rings and $(viii)$, partial orders.

In the example (vi), the class of torsion R-modules is not a first order class as we have to say $(\forall x) \bigvee\limits_{r \in R \setminus \{0\}} rx = 0$ and we really need to use an infinite disjunction.

The situation is similar for the class of nil rings (example (iv)). In example (vii), the class of rings with $\leq_{\mathfrak{K}}$ defined using finitely generated subrings not only is the class of structures not elementary but $\leq_{\mathfrak{K}}$ is neither $\prec$ nor $\subseteq$. In the example (vii), R-modules, K is elementary but $\leq_{\mathfrak{K}}$ is different.

Recall[6] the traditional frame of model theory are the so-called elementary (or first order) classes. That is, for some vocabulary τ, and set T of so-called sentences in first order logic in this vocabulary, $K = \mathrm{Mod}_T = \{M : M \text{ a } \tau\text{-structure satisfying every sentence of } T\}$ and $\leq_{\mathfrak{K}}$ being $\prec$, "elementary submodel". Recall that $M \prec N$ if $M \subseteq N$ and for every first order formula $\varphi(x_0, \ldots, x_{n-1})$ in the (common) vocabulary, i.e., from the language $\mathbb{L}(\tau)$ and $a_0, \ldots, a_{n-1} \in M$, $\varphi(a_0, \ldots, a_{n-1})$ is satisfied by M, (symbolically $M \models \varphi[a_0, \ldots, a_{n-1}]$) <u>iff</u> N satisfies this.

Now here an elementary class is one of the form $(\mathrm{Mod}_T, \prec)$, any such class is an abstract elementary class (see below). A different abstract elementary class derived from T is $(\mathrm{Mod}_T, \subseteq)$ but then we should restrict ourselves to T being a set of universal sentences or just Π_2-sentences as we like to have closure under direct limits. For each such T another abstract elementary class which can be derived from it is $(\{M \in \mathrm{Mod}_T : M \text{ is existentially closed}\}, \subseteq)$.

We are <u>not disputing</u> the choice of first order classes as <u>central</u> in model theory but there are many interesting other classes. Most notably for algebraists are classes of locally finite structures and for model theorists are $(\mathrm{Mod}_\psi, \prec_{\mathscr{L}})$ where ψ belongs to the logic denoted by $\mathbb{L}_{\omega_1, \omega}(\tau)$ or just $\psi \in \mathbb{L}_{\lambda^+, \omega}(\tau)$ for some λ where $\mathscr{L}$ is a fragment of this logic to which the sentence ψ belongs; if $\psi \in \mathbb{L}_{\omega_1, \omega}(\tau)$ we may choose a countable such $\mathscr{L}$.

(This logic may seem obscure to non-logicians but it just means that we allow to say $\bigwedge\limits_{i \in I} \varphi_i(x_0, \ldots, x_{n-1})$ where I has at most λ members

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[6]we urge the logically challenged: when lost, jump ahead

so enable us to say "a ring is nill, locally finite, etc.", <u>but</u> not "$<$ is a well ordering").

In some sense if we look at classification theory of elementary classes as a building, we note that several "first floors" disappear (in the context of abstract elementary class) but we aim at saving considerable part of the rest (of course not all) by developing a replacement for those lower floors.

We may put in the basement the downward LS theorem (there are small $N \prec M$); it survives. But not so the compactness theorem even very weak forms like "if $\bar{a} = \langle a_n : n \in \mathbb{N} \rangle, \bar{b} = \langle b_n : n \in \mathbb{N} \rangle$ are sequences of members of M and f_n is an automorphism of M mapping $\bar{a} \upharpoonright n$ to $\bar{b} \upharpoonright n$ then some $\leq_\mathfrak{K}$-extension of M has an automorphism mapping $\bar{a}$ to $\bar{b}$" do not hold in arbitrary a.e.c. (Note that for "(D, λ)-homogeneous models" (e.g. [Sh 3]) such forms of compactness hold. The point of [Sh 394] is to start investigating classes for which all is nice except that types are not determined by their small restrictions, that is, defining $\mathbb{E}_N^\kappa = \{(p, q) : p, q \in \mathscr{S}(N) \text{ and } M \in K_\kappa \Rightarrow p \upharpoonright M = q \upharpoonright M\}$, this is, a priori, not the equality ([Sh 394, 1.8,1.9,pg.4]). We lose as well the upward LST theorem (every model has a proper $<_\mathfrak{K}$-extension); (those fit the first floor).

Also in abstract elementary classes the roles of formulas disappear. Hence we lose the notion of the type of an element a over a set A inside a model M; so the second floor including the "κ-saturated model" (in the traditional sense) goes down the drain as the types disappear.

What is saved? (I.e. not by definitions but in the positive case of a dividing line which has a non-structure result.) In a suitable sense, we save: non-forking amalgamation of models, prime models, a decomposition of a model over a non-forking tree of models (a relative of free amalgamation), and for a different notion of type, being (saturated and) orthogonal, regular and eventually the main gap for the parallel of $\aleph_\varepsilon$-saturated model of a superstable T.

We now try to describe our aim in broad terms; if this seems vague, in (B) below we describe it in a restricted case more concretely. Our aim is to consider a family of classes $\mathfrak{K}$ (all the "reasonable" classes) and try to <u>classify</u> them in the sense of taxonomy, we look for <u>dividing lines</u> among them. This means dividing the family to two, one part are those which are "high", "complicated". Typically we have for each $\mathfrak{K}$ in the "high side" a <u>non-structure</u> result, saying there are many complicated such models $M \in K$ (in suitable sense). Those in the other side, the "low" one have some "positive" theory, we have to some extent understood those models, e.g.

they have a good dimension theory.

A reader interested to see more quickly what is done rather than why it is done and what are our hopes should go to (C) below.

A good dividing line of a family of classes is such that we really can say something on both sides, ideally it also should help us prove things on all K's by division to cases. So it seems advisable to prove the equivalence of an external property (like not having many models) and an internal property (some understanding of models of K). Now clearly such a dividing line is interesting but, of course, there are properties which are interesting for other reasons. (See more on this in the end of (A) of §1).

(B) The structure/non-structure dichotomy

More specifically we may ask: which classes have a structure theory? By a structure theory we mean "determined up to isomorphism by an invariant called the dimension or several dimensions or something like that". A non-structure property (or theorem) will be a strong witness that there is no structure theory. So the question is:

2.5 Question: When does a class $\mathfrak{K}$ of models have a structure theory? In particular, each model from $\mathfrak{K}$ is characterized up to isomorphism by a "complete set of reasonable invariants" like those of Steinitz (for algebraically closed fields) and Ulm (for countable torsion abelian groups).

This is still quite vague, and it takes some explanation (and choices) to make it concrete. Instead we shall be even more specific. We shall explain two more concrete questions: categoricity and the main gap and the solution in the known (first order countable vocabulary) case. Counting the number of models in a class seems very natural and to make sense we have to count them in each cardinality separately. If the reader is not enthusiastic about this counting, some alternative questions lead us to the same place: e.g.: having models which are almost isomorphic but not really isomorphic (see more in $(*)_2$ from §1(B)(c)).

2.6 Definition. For a class K of models and infinite cardinal λ let $\dot{I}(\lambda, K)$ be the number of models in K of cardinality λ up to isomorphism. So for any K it is a function from Card, the class of cardinals to itself; we may write $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ instead of K.

Now a priori we may get quite arbitrary functions. But it seems reasonable to hope that all our classes $\mathfrak{K}$ will have a simple function $\lambda \mapsto \dot{I}(\lambda, \mathfrak{K})$ and classes with a "structure theory" will have such functions with small values. It seems more hopeful to try to first investigate the most extreme cases (being one and being maximal), considering both our chances to solve and for getting an interesting answer; also we expect the "upper" one to give the important dividing lines. It is most natural to start asking about the spectrum of existence, i.e., being non-zero, i.e., what can

be $\{\lambda : \mathfrak{K}_\lambda \neq \emptyset\}$? This had been answered quite satisfactorily (see I.?,I.? above LST($\mathfrak{K}$), it is an initial segment with a known bound), and it seems easier at least from the present perspective.

Considering this, the number one naturally has a place of honor; this is categoricity. Recall K is said to be categorical in λ iff $\dot{I}(\lambda, K) = 1$.

A natural thesis is

2.7 Thesis: If we really understand when a (reasonable) class is categorical in λ it should have little dependence on λ, ignoring "few, exceptional" cardinals.
[Why? How can we understand why $\mathfrak{K}$ is categorical in λ? We should know so much on the class so that given two models from K of cardinality λ we can construct in a coherent way an isomorphism from one onto the other; but this should work for any other (large enough) cardinal. Also being categorical implies the model is a very simple one, analyzable.

This is, of course, not true for every class of, e.g. if K is the class of $\{(I, <):$ $<$ well orders I and if $|I|$ is a successor cardinal then every initial segment has cardinality $< |I|\}$. This class is categorical in $\aleph_\alpha$ iff $\aleph_\alpha$ is a limit cardinal (we could change it to "α even", etc). However, we have to restrict ourselves to "reasonable" classes.]

An antagonist argument against the thesis 2.7 is that for first order T, the class $\{\lambda : T$ has in λ a rigid model, i.e., one without (non-trivial) automorphism$\}$, e.g. can be "any class of cardinals" in some sense, e.g., $\{\aleph_3, \aleph_{762}, \beth_{\omega_3}$, first inaccessibly cardinality$\}$. This class may be, essentially, any Σ_2^1 class of cardinals (see [Sh 56]).

We may answer that rigidity implies a complicated model so we may have T coding a definition of a complicated class, of cardinals, whereas being categorical implies the models are simple. The antagonist may answer that allowing enough classes of models it would not work, the categoricity spectrum will be weird and probably Łos (see below) has no good enough reasons for his conjecture (of course we can argue till the problem is resolved). We may answer that Łos conjecture implicitly says that first order classes (of countable vocabulary) are "nice", "analyzable". So 2.7 begs the question of which classes are reasonable and this book contend that abstract elementary classes are.

Of course, there may be reasonable classes for which "$\mathfrak{K}$ is categorical" depends on simple properties of the cardinal (e.g. being strong limit).

More specifically we may ask: is it true for every (relelvant) $\mathfrak{K}$, either $\mathfrak{K}$ is categorical in almost every λ or non-categorical in almost every λ? Indeed Łos had conjectured that if an elementary class $\mathfrak{K}$ with countable vocabulary is categorical in one $\lambda > \aleph_0$ then $\mathfrak{K}$ is categorical in every $\lambda > \aleph_0$, having in mind the example of algebraically closed fields of a fixed characteristic. A milestone in mathematical logic history was Morley's proof of this conjecture. The solution forces you to understand such $\mathfrak{K}$.

We may ask: Is $\dot{I}(\lambda, \mathfrak{K})$ a non-decreasing function? Of course, this is a question

on K but the assumptions are on $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$. This sounds very reasonable as "having more space we have more possibilities". For elementary $\mathfrak{K}$ with countable vocabulary this was conjectured by Morley (for $\lambda > \aleph_0$). It is not clear how to prove it directly so it seemed to me a reasonable strategy is to find some relevant dividing lines: the complicated classes will have the maximal number of models, the less-complicated ones can be investigated as we understand them better. This may lead us to look at the dual to categoricity, the other extreme - when $\dot{I}(\lambda, T)$ is maximal (or just very large).

2.8 Definition. The main gap conjecture for K says that either $\dot{I}(\lambda, K)$ is maximal (or at least large) for almost all λ <u>or</u> the number is much smaller for almost all λ; for definiteness we choose to interpret "almost all λ" as for every λ large enough.

(We cheat a little: see 2.10).

This seems to me preferable to "$\dot{I}(\lambda, K)$ is non-decreasing" being more robust; this will be even more convincing if we succeed in proving the stronger statement:

<u>2.9 The structure/non-structure Thesis</u> For every reasonable class either its models have a complete set of cardinal invariants <u>or</u> its models are too complicated to have such invariants.

This had been accomplished for elementary classes (= first order theories) with countable vocabularies. We suggest that the main gap problem is closely connected to 2.9.

So ideally, for classes $\mathfrak{K}$ with structure for every model M of $\mathfrak{K}$ we should be able to find a set of invariants which is complete, i.e., determines M up to isomorphism. Such an invariant is the isomorphism type, so we should restrict ourselves to more reasonable ones, and the natural candidates are cardinal invariants or reasonable generalizations of them. E.g. for a vector space over $\mathbb{Q}$ we need one cardinal (the dimension = the cardinality of any basis). For a vector space over an algebraically closed field, two cardinals; (the dimension of the vector space and the transcedence degree (= maximal number of algebraically independent elements) of the field, both can be any cardinal; of course, we have also to say what the characteristic of the field is). For a divisible abelian group G, countably many cardinals (the dimension of $\{x \in G : px = 0\}$ for each prime p and the rank of $G/\mathrm{Tor}(G)$ where $\mathrm{Tor}(G)$ is the subgroup consisting of the torsion members of G, i.e. $\{x \in G : nx = 0$ for some $n > 0\}$). For a structure with countably many one-place relations P_n (i.e., distinguished subsets), we need $2^{\aleph_0}$ cardinals (the cardinality of each intersection of the form $\cap\{P_n^M : n \in u\} \cap \{M \backslash P_n^M : n \notin u\}$) for u a set of natural numbers).

We believe the reader will agree that every structure of the form $(|M|, E)$, where E is an equivalence relation, has a reasonably complete set of invariants: namely, the function saying, for each cardinal λ, how many equivalence classes of this cardinality occur. Also, if we enrich M by additional relations which relate only

E-equivalent members and such that each E-equivalence class becomes a structure with a complete set of invariants, then the resulting model will have a complete set of invariants. We know that even if we allow such generalized cardinal invariants, we cannot have such a structure theory for every relevant class (e.g. the class of linear orders has no such cardinal invariants). So if we have a real dichotomy as we hope for, we should have a solution of (a case of) the main gap conjecture which says each class K either has such invariant or is provably more complicated.

Let us try to explicate this matter. We define what is a λ-value of depth α by induction on the ordinal α: for $\alpha = 0$ it is a cardinal $\leq \lambda$, for $\alpha = \beta + 1$ it is a sequence of length $\leq 2^{\aleph_0}$ of functions from the set of λ-values of depth β to the set of cardinals $\leq \lambda$ or a λ-value of depth β, and for α a limit ordinal it is a λ-value of some depth $< \alpha$.

An invariant [of depth α] for models of T is a function giving, for every model M of T of cardinality λ, some λ-value [of depth α] which depends only on the isomorphism type of M. If we do not restrict α, the set of possible values of the invariants is known, in some sense, to be as complicated as the set of all models.

This leads to:

<u>2.10 Main Gap Thesis</u>: 1) A class K has a structure theory if there are an ordinal α and invariants (or sets of invariants) of depth α which determines every structure (from K) up to isomorphism.
2) If K fails to have a structure theory it should have "many" models and we expect to have reasonably definable such invariants.

We can prove easily, by induction on the ordinal α, that

2.11 Observation. The number of $\aleph_\gamma$-values of depth α has a bound $\beth_\alpha(|\tau_K| + |\gamma|)$ where

$$\beth_\beta(\mu) = \mu + \prod_{\varepsilon < \beta} 2^{\beth_\varepsilon(\mu)}.$$

2.12 Corollary of the thesis. *If $\mathfrak{K}$ has a structure theory by the interpretation of 2.10 <u>then</u> there is an ordinal α such that for every ordinal γ, $\mathfrak{K}$ has $\leq \beth_\alpha(|\tau_\mathfrak{K}| + |\gamma|)$ non-isomorphic models of cardinality $\aleph_\gamma$.*

It is easy to show, assuming e.g., the G.C.H., that for every α there are many γ's such that $\beth_\alpha(|\omega + \gamma|) < 2^{\aleph_\gamma}$ and even $< \aleph_\gamma$. Thus, if one is

able to show that $\mathfrak{K}$ has $2^{\aleph_\gamma}$ models of cardinality $\aleph_\gamma$, this establishes non-structure.

In the case in which the main gap was proved, it turns out that there are only a few "reasons" for an elementary class $\mathfrak{K}$ with countable vocabulary to have the maximal number of models:

(a) $\mathfrak{K}$ is so called unstable, prototypical example are the class of infinite linear orders and the class of random graphs [formally: in some model from $\mathfrak{K}$ some first order formula $\varphi(\bar{x}, \bar{y})$ with $\ell g(\bar{x}) = m = \ell(\bar{y})$ for every linear order I there is $M \in \mathfrak{K}$ and an m-tuple $\bar{a}_t$ from M for each $t \in I$ such that $\varphi[\bar{a}_s, \bar{a}_t]$ is satisfied in M iff $s <_I t$]

(b) $\mathfrak{K}$ has the so called OTOP, it is similar to (a), but the order is defined in a different way, not by a so-called first order formula but by a formula of the form $(\exists \bar{z}) \bigwedge_n \varphi_n(\bar{x}, \bar{y}, \bar{r})$. The prototypical example is straightforward but somewhat cumbersome.

(c) it has the DOP, this is harder to define. An easy example is: two cross-cutting equivalence relations. It means that in some members M of $\mathfrak{K}$, we can define large linear orders by using dimensions

A proto-typical example is: for some infinite I and $R \subseteq I \times I$, $M_{I,R}$ has universe $I \cup \{(s, t, \alpha) : s \in I, t \in I, \alpha < \omega_1 \text{ and } (s, t) \in R \Rightarrow \alpha < \omega\}$ and relation $P^M = \{(s, t, a) : a = (s, t, \alpha) \text{ for some } \alpha\}$. So R can be defined in $M_{I,R}$ (though is not a relation of M) as $\{(s, t)$: the set $\{x : M_{I,R} \models P(s, t, x)\}$ is uncountable$\}$. But the definition is not first order, it speaks on dimension (actually we can also interpret any graphs). Note that $T = \mathrm{Th}(M_{I,R})$ does not depend on R.

(d) $\mathfrak{K}$ is so called unsuperstable; proto-typical example $({}^\omega I, E_n)_{n<\omega}$ where ${}^\omega I$ is the set of functions from $\mathbb{N}$ into I and $E_n = \{(\eta, \nu) : \eta, \nu \in {}^\omega I$ and $\eta \upharpoonright n = \nu \upharpoonright n\}$

(e) T is deep, proto-typical example is the class of graphs which are trees (i.e. with no cycles).

We return to the more concrete question: the main gap and the thesis 2.9. We can hope that a non-structure theorem should imply $\dot{I}(\lambda, K)$ is large, whereas a structure theorem should enable us to show it is small and even allow us to show it is non-decreasing, and to compute it.

Actually the picture of the "non-structure" side (in the resolved case) is more complicated. In some classes "reasons" (a)-(d) fail but

"reason" (e) holds. In this case the members of $\mathfrak{K}$ are essentially as complicated as graphs which are trees (i.e., no cycle); for them we get the maximal number of non-isomorphic models, but we have a "handle" on understanding the models. The following result illustrates this kind of understanding: possibly[7] for some λ we cannot find λ models (of any cardinality) no one embeddable into the others. If one of clauses (a)-(d) holds, there are stronger results in the inverse direction (e.g. we can code stationary sets modulo the club filter). So it seemed that we end up with a trichotomy rather than a dichotomy. That is, for the question of counting the number of models up to isomorphism the middle family behaves more like the high one: has maximal number. But for the question mentioned above and also for questions of the form: "are there two very similar non-isomorphic models in the class" the middle family behaves like the low (e.g. we can build reasonable invariants when not restricting the ordinal depth). Still there are clear results for each of the three families.

It was (and is) our belief that there is such a theory even for abstract elementary classes and that we should look at what occurs at large enough cardinals, as in small cardinals various "incidental" facts interfere. Notice that a priori there need not be a solution to the structure/non-structure problem or to the spectrum of categoricity problem: maybe $\dot{I}(\lambda, T)$ can be any one of a family of complicated functions, or, worse, maybe we cannot characterize reasonably those functions, or, maybe the question of which functions occur is independent of the usual axioms of set theory.

Now, of course, the aim of classification is not just those specific questions. We rather think and hope that trying to solve them will on the way give interesting dividing lines among the classes. A class K here may have too many models but still we can say much on the structure of its models.

Now the thesies underlining the above is

2.13 Thesis

(a) dividing lines are interesting, and obviously reasonable test questions are a good way to find them (and we try to use test questions of self-interest)

(b) good dividing lines throw light also on questions which seem very different from the original test questions

(c) in particular, investigating $\dot{I}(\lambda, K)$ (and more profoundly, characterizing the classes with complete set of invariants) is a good way to find interesting dividing lines, but naturally there are other ways to arrive at them and

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[7]formally: if some (mild) large cardinal exists

(d) there are measures of complexity of a class (other than $\dot{I}(\lambda, K)$) which lead to interesting dividing lines and some such work was done on elementary classes (see §1).

Behind the discussion above also stands

2.14 Thesis: To investigate classes K it is illuminating to look for each λ, at problems on $\mathfrak{K}_\lambda, \mathfrak{K}$ which is restricted to cardinal λ and

(a) to try to prove that the answer does not depend on λ or at least depends just on a small amount of information on λ

(b) to discard too small cardinals (essentially to look at asymptotic behaviour)

This seems to be successful in discovering stability (and superstability).

> An illustration is that Rowbottom had defined λ-stable (i.e. $A \subseteq M \wedge |A| = \lambda \Rightarrow |\mathscr{S}(A, M)| \leq \lambda$) but it seems to me only having ([Sh 1]) the characterization of $\{\lambda : T$ stable in $\lambda\}$ and the equivalence with the order property and defining "T stable" started stability theory. (Of course, for his aims this was irrelevant).

The rationale is that if the answer is the same for "most λ", this points to a profound property of the class and it forces you to find inherent principles which you may not be so directly led to otherwise. Hence it probably will be interesting even if you care little about these cardinals. A parallel may be that even low dimension algebraic topologists were interested in the solution of Poincare conjecture for dimension ≥ 5. Also the behaviour in too small cardinals may be "incidental". So the class of dense linear order with neither first nor last element and the class of atomless Boolean Algebra or the class of random enough graphs are categorical in $\aleph_0$, but have many complicated models in higher ones. (One may feel these are low theories. This is true by some other criterions, other test problems; in fact, there are dividing lines among the elementary classes for which they are low. Still, for the test questions considered here, provably those classes are complicated, e.g., in a strong sense do not have a set of cardinal invariants characterizing the isomorphism type).

You may wonder:

2.15 Question: Do we recommend dividing lines everywhere? (in mathematics) or is this something special for model theory?

Now dividing lines are meaningful in many circumstances. But on the one hand it is better to list all simple finite groups than to find a dividing line among them. Similarly for the elementary classes categorical in every $\lambda \geq \aleph_0$. On the other hand, surely for many directions there are no fruitful dividing lines. The thesis that appeared here means that for broad front in model theory this is fruitful. (Not

everywhere: too strong infinitary logics are out). It seemed that this has been vindicated for stability (and to some extent for simplicity and hopefully for (the family of) dependent elementary classes).

It may be helpful to compare this to alternative approaches in model theory. One extreme position will say that there is a central core in mathematics (built around classical analysis and geometry; and number theory of course) and other areas have to justify themselves by contributing something to this central core. Dealing with cardinals is pointless bad taste, and while some interaction of elementary classes with cardinals had been helpful, its time has passed.

It seemed to me that the criterion and its application leave out worthwhile directions. We all know that some neighboring subjects are just hollow noise and sometimes we are even right. So an excellent witness for a mathematical theory to be worthwhile is its ability to solve problems from others, preferably classical areas or problem from other sciences. Certainly a sufficient condition. What is doubtful is whether it is a necessary condition; we do not agree.

However, even within this narrow criterion, the direct attack is not the only way to look for applications to other areas. Not so seldom do we find that only after developing strong enough theory, deep applications become possible, the history of model theory seems to support this (in particular, lately in works of Hrushovski and Zilber). Looking at large enough cardinals serve as asymptotic behaviour, in which it is more transparent what are the general outlines of the picture.

The reader may wonder how this work is related, e.g. to category theory? universal algebra? soft model theory?. For category theory this work, in short, is closer than classical model theory but still not really close, similarly in category theory each class $\mathfrak{K}$ is equipped with a notion of mapping (rather than $\leq_{\mathfrak{K}}$ being defined from K by some specific logic as in classical model theory). But here we restrict ourselves to embeddings (this is not unavoidable but things are already hard enough without this) and the main difference is that we do not forget the elements.

What about universal algebra? A traditional model[8] theorist definition of model theory is combining universal algebra and logic, so a large part of this work is, by that definition, in universal algebra. I do not see any reason to disagree but still the methods and results are well rooted in the model theoretic tradition.

What about soft model theory? Though our work itself does not need soft model theory, it fits well there (and Chapter I, Chapter IV use infinitary logics hence are not discussed in this part).

> First, for many important logics $\mathscr{L}$, for theories $T \subseteq \mathscr{L}(\tau)$ the class $(\mathrm{Mod}_T, \prec_{\mathscr{L}(\tau)})$ or variants are abstract elementary classes (certainly for the logic $\mathbb{L}_{\lambda^+,\omega}$) and by choosing the $\leq_{\mathfrak{K}}$ appropriately also $\mathbb{L}(\mathbf{Q}^{\mathrm{card}}_{\geq\lambda})$; in fact they were the original motivation to look at abstract

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[8]but no universal algebraist agree

elementary classes. So if you ask for the part of soft model theory
dealing with classification theory or at least investigate categoricity,
you arrive here. Also not just varying the logic, but fixing a class
Mod_T fits it well.

This work certainly reflects the author's preference to find something in the
white part of our map, the "terra incognita" rather than understand perfectly
what we have reasonably understood to begin with (which is exemplified by looking
at abstract elementary classes on which our maps reflect our having little to say
on them, rather than FMR theories or o-minimal theories, cases where we had
considerable knowledge and would like to complete it). Anyhow, by experience,
there will not be many complaints on lack of generality and broadness.

Note that we would like to get results, not consistency results and allowing
definability of well ordering or completeness runs into set-theoretic independence
results so restricting ourselves to an abtract elementary class, a framework which
excludes well ordering and complete spaces is reasonable. But we shall not really
object to cardinal arithmetic assumptions like weak forms of GCH.

> In fact, having the <u>non-structure</u> results depend on the universe
> of set theories is not desirable but is reasonable, as they still witness
> the impossibility of a positive theory. It is reasonable to adopt this
> as part of the rules of the games. In some cases, consistency results
> forbid us to go further (see, e.g. [Sh:93]). But still the positive side
> should better be in ZFC.

(C) <u>Abstract elementary classes</u>

We now return to the question: With which classes of structures we shall deal?
Obviously, "a class of structures" is too general. Getting down to business we
concentrate on

$\boxtimes$ (a) abstract elementary classes

(b) good λ-frames

(c) beautiful λ-frames.

In short, in $\boxtimes(a)$, see below, we suggest abstract elementary classes (a.e.c.) as
our framework, i.e., the family of classes we try to classify; it clearly covers much
ground and seems, at least to me, very natural. What needs justification is whether
we can say on it interesting things, have non-trivial theorems.

Among elementary (= first order) classes we know which classes have reason-
able dimension theory, the so called superstable elementary classes; and we like
to understand the case for the family of abstract elementary class . In $\boxtimes(c)$, see

below in §3(C), we suggest beautiful λ-frames as our "promised land", as a context where we have reasonable understanding, e.g., have dimension theory, can prove the main gap, etc. (but of course more wide families "on our way" probably will be interesting per se). Now it is very unsurprising that if we assume enough axioms, we shall regain paradise (which means here quite full fledged analog to the so called superstability theory, at least for my taste). Hence the problem in justifying the choice of $\boxtimes(c)$ is mainly not in pointing to many good properties but to show that there are enough such frames and it helps prove theorems not mentioning it. On the second (i.e. prove theorems not mentioning them...), see e.g. 2.20(2) below. In our context ideally the first (i.e. "there are enough such frames...") means to show that they are the only ones, i.e., the broadest family of abstract elementary class which has such a good dimension theory. We are far from this, still we would according to our "guidelines" like at least to get beautiful frames by choosing to consider the classes which fall on the "low" side (in the elementary classes case) by dividing lines (= dichotomies) inside a family of classes which is large and natural, here among abstract elementary classes. That is, the program is to suggest some dividing lines, for the high side to prove the so-called non-structure theorems and for the low side to have some theory. Being always in the low sides we should arrive to beautiful frames.

But most of our work falls under $\boxtimes(b)$, good λ-frames. So it needs double justification: on the one hand we have to show it arises naturally from our program. [In detail, a weak case for "arising naturally" is to start with an abstract elementary classes satisfying some external condition of being "low" like categoricity, and prove that "inside $\mathfrak{K}$" we can find good frames. A strong case is to find a dividing line such that for each low $\mathfrak{K}$ we can find inside it "enough" good frames, and for all other "few". There is another meaning of "arising naturally" which would mean that we have looked at some natural examples and extracted the definition from their common properties; this is not what we mean. We rather try to solve questions on the number of models but of course the first order case was before our eyes as first approximation to the paradise we would like to arrive to.]

On the other hand for such frames, possibly with more assumptions justified similarly we can say something significant.

In fact, we see good λ-frames essentially as the rock-bottom analogs of the family of elementary classes called superstable mentioned above.

We shall discuss $\boxtimes(a)$ and (b) and (c) in more detail. We start with

$\boxtimes(a)$ abstract elementary classes.

Recall the definition of abstract elementary classes Definition 2.2.

2.16 Explanation: An abstract elementary class is easy to explain (probably much

simpler than elementary (= first order) class). Such $\mathfrak{K}$ consists of a class K of structures = models, all of the same "kind", e.g. all rings have the same kind, but a group has a different kind. We express this by saying "all members of K has the same vocabulary $\tau = \tau_K$". E.g., K consists of objects of the form $M = (A^M, F_0^M, F_1^M, Q^M), A^M$ its universe, a non-empty set, F_ℓ^M a binary function on it, Q^M a binary relation. $\mathfrak{K}$ has also an order $\leq_{\mathfrak{K}}$ on K, its notion of being a sub-structure (which refines the standard notion). Now $(K, \leq_{\mathfrak{K}})$ have to satisfy some requirements: preservation under isomorphisms, $\leq_{\mathfrak{K}}$ being an order, preserved by direct limits and also direct limits inside $N \in K$, remembering that our mapping are embedding. Also if $M_1 \subseteq M_2$ are both $\leq_{\mathfrak{K}}$-sub-structures of N then $M_1 \leq_{\mathfrak{K}} M_2$, and lastly we demand every $M \in K$ has a countable $\leq_{\mathfrak{K}}$-sub-structure including any pregiven countable set of elements (or replace countable by a fix cardinality, we ignore this point in the introduction; see II§1).

Concerning "$M_\ell \leq_{\mathfrak{K}} N, (\ell = 1, 2), M_1 \subseteq M_2 \Rightarrow M_1 \leq_{\mathfrak{K}} M_2$" note that if we define $\leq_{\mathfrak{K}}$ as $\prec_{\mathscr{L}}$ for any logic, this will hold.

For elementary classes $\mathfrak{K}$, because of the so-called compactness and Löwenheim-Skolem-Tarski theorems, the situation in all cardinals is to a significant extent similar.

In particular, if $\mathfrak{K}$ is an elementary class (with countable vocabulary) and λ_1, λ_2 are (infinite) cardinals then there is $M \in \mathfrak{K}$ of cardinality λ_1 iff there is $M \in \mathfrak{K}$ of cardinality λ_2. So recalling that $K_\lambda = \{M \in K : M$ has cardinality $\lambda\}$ and $\mathfrak{K}_\lambda = (K_\lambda, \leq_{\mathfrak{K}} \restriction K_\lambda)$ we have $K_{\lambda_1} \neq \emptyset \Leftrightarrow K_{\lambda_2} \neq \emptyset$. Moreover, any infinite $M \in \mathfrak{K}$ has $\leq_{\mathfrak{K}}$-extension in every larger cardinality. But for abstract elementary classes it is not necessarily true, and even if $(\forall \lambda) K_\lambda \neq \emptyset$ there may be many $\leq_{\mathfrak{K}}$-maximal models, i.e., $M \in K$ such that $M \leq_{\mathfrak{K}} N \Rightarrow M = N$. This (and more) makes the theory very different.

The context of abstract elementary class may seem so general, we may doubt if anything interesting can be said about it; still note that this context does not allow the class of Banach spaces as the union of an increasing chain is not necessarily complete. Certainly a loss. Also the class $(W, \subseteq)$, the class of well orders, is not an abstract elementary class ; (recall I is a well order if it is a linear order such that every non-empty set has a first element). Similarly the class $(K^{\mathrm{fgi}}, \subseteq)$ where $K^{\mathrm{fgi}} = $ the class of rings (or even integral domains) in which every ideal is finitely generated, is not an abstract elementary class (where $\leq_{\mathfrak{K}}$ is being a subring). However, we get an abstract elementary class when we consider only $K_{\leq n} = $ the class of rings in which every ideal is generated by $\leq n$ elements.

We may like to replace n by a countable ordinal α, i.e., $K^{\mathrm{fgi}}_{\leq \alpha} = \{M \in$

$K : \mathrm{dp}_M(\emptyset) \leq \alpha\}$; where for a ring M we define dp:$\{u : u \subseteq M$ finite$\} \to$ the ordinals by $\mathrm{dp}_M(u) = \cup\{\mathrm{dp}(w) + 1 : u \subseteq w$ and w is not included in the ideal of M which u generates$\}$. But then we have problems with closure under unions; a reasonable remedy is to have an appropriate $\leq_{\mathfrak{K}}$: $M \leq_{\mathfrak{K}} N$ if M, N are rings and for every finite $u \subseteq M$ we have $\mathrm{dp}_N(u) = \mathrm{dp}_M(u)$.

Why have we restricted ourselves to "countable α"? Only because in clause (g) of Definition 2.2 we have used "countable".

But the family of abstract elementary classes includes all the examples listed in 2.1 in the beginning (of this section, 2).

Also, other abstract elementary classes are $(K, \prec)$ where K is the class of locally finite models of a first order theory T. Another example is $(\mathrm{Mod}_\psi, \prec_{\mathscr{L}})$ where ψ is a sentence from logic $\mathbb{L}_{\lambda^+,\omega}$ with $\mathscr{L}$ the set of subformulas of ψ. Also $(K, \prec)$ where $P \in \tau_{\mathfrak{K}}$ is a unary predicate, T first order and $K = \{M \in \mathrm{Mod}_T : P^M = \mathbb{N},$ the natural numbers$\}$.

A natural property to consider is amalgamation. We say that $\mathfrak{K}$ has the amalgamation property when for any $M_\ell \in \mathfrak{K}, \ell = 0, 1, 2$ and $\leq_{\mathfrak{K}}$-embedding f_1, f_2 of M_0 into M_1, M_2 respectively (this means that f_ℓ is an isomorphism from M_0 onto some $M'_\ell \leq_{\mathfrak{K}} M_\ell$) there are $M_3 \in \mathfrak{K}$ and $\leq_{\mathfrak{K}}$-embeddings g_1, g_2 of M_1, M_2 into M_3 respectively such that $g_1 \circ f_1 = g_2 \circ f_2$. Should we adopt it? Now it is a very important property, we would like to have it, but it is a strong restriction (our prototyical problem, models of $\psi \in \mathbb{L}_{\omega_1,\omega}$ fails it); so we do not assume it, but it will appear as a dividing line.

So the thesis is

2.17 Thesis:

(a) In the context of abstract elementary classes we can answer some non-trivial questions

(b) In particular we can say something on the categoricity spectrum

(c) In the long run a parallel to the main gap will be found.

A reasonable reader may require an example of results. First we quote [Sh 576] represented here in [Sh:E46]:

2.18 Theorem. *Assume* $2^{\aleph_\alpha} < 2^{\aleph_{\alpha+1}} < 2^{\aleph_{\alpha+2}}$ *and* $\mathfrak{K}$ *is an abstract elementary class categorical in* $\aleph_\alpha$, *in* $\aleph_{\alpha+1}$ *and has an "intermediate" number of models in* $\aleph_{\alpha+2}$, *<u>then</u>* $\mathfrak{K}$ *has at least one model in* $\aleph_{\alpha+3}$.

Note that

2.19 Notation. If $\lambda = \aleph_\alpha$ we let $\lambda^{+n} = \aleph_{\alpha+n}$, so can write this theorem in such a notation, similarly later.

So it is an example for 2.17(a)+(b): not "every function" can occur as $\lambda \mapsto \dot{I}(\lambda, \mathfrak{K})$.

Note that this theorem gives a weak conclusion, but with very weak assumptions. In fact at first glance it seems we are facing a wall: our assumptions are so weak to exclude all possible relevant methods of model theory, in particular all relatives of compactness.

> I.e., we have no compact (even just $\aleph_0$-compact) logic defining our class. Of course, the upward LST cannot be used, it does not make sense: the desired conclusion is a weak form of it. As for the downward Löwenheim Skolem-Tarski theorem, with only three cardinals available it seems to say very little.
>
> We do not have formulas hence no types and no saturated models. Here we cannot use versions of "well ordering is undefinable" as in previous cases (see Chapter I; if $\aleph_\alpha = \aleph_0$ and $\mathfrak{K}$ is reasonable we have used "no $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ defines well ordering (in a richer vocabulary)"; this does not apply in [Sh 576], i.e. [Sh:E46], even when $\lambda = \aleph_0$, as we demand only $\mathrm{LST}(\mathfrak{K}) \leq \aleph_0$ rather than "$\mathfrak{K}$ is a $\mathrm{PC}_{\aleph_0}$-class"; and we certainly like to allow any $\aleph_\alpha$). Also in general we cannot find Ehrenfeuch-Mostowski models (another way to say well orders are not definable). Also we do not assume the existence of relevant so called large cardinals, e.g. $\mathfrak{K}$ is definable in some $\mathbb{L}_{\kappa,\omega}, \kappa$ a compact or just a measurable cardinal. So indeed no remnants of compactness are available here.

The proof of 2.18 leads us to our second framework, good λ-frames, which has a crucial role in our investigations, see below. The main neatly stated result in Chapter II (part (1) of 2.20), Chapter III(part (2) of 2.20) is:

> (omitting a weak set theoretic assumption which will be eliminated in the full version of [Sh 838]).

2.20 Theorem. *Assume $\mathfrak{K}$ is an abstract elementary class .*
1) $\mathfrak{K}$ has a member in $\aleph_{\alpha+n+1}$ <u>if</u> ($n \in \mathbb{N}$ and)

   (a) $n \geq 2$ and $2^{\aleph_\alpha} < 2^{\aleph_{\alpha+1}} < \ldots < 2^{\aleph_{\alpha+n}}$

   (b) $\mathfrak{K}$ is categorical in $\aleph_\alpha$ and in $\aleph_{\alpha+1}$

   (c) $\mathfrak{K}$ has a model in $\aleph_{\alpha+2}$

(d) $\dot{I}(\aleph_{\alpha+m}, \mathfrak{K})$ *is not too large for* $m = 2, \ldots, n$.

2) *If (a)-(d) holds for every n* <u>*then*</u> $\mathfrak{K}$ *is categorical in every* $\aleph_\beta \geq \aleph_\alpha$.

> Actually in this theorem "$\mathfrak{K}$ having Löwenheim-Skolem-Tarski number $\leq \lambda$" (rather than $\aleph_0$) is enough.

(D) <u>Toward Good λ-frames (i.e. $\boxtimes(b)$</u>:

<u>2.21 Thesis</u> Good λ-frames are a right context to start our "positive" structure theory.

> They are a rock-bottom parallel of superstable elementary classes.

Now compared to abstract elementary classes, much more has to be said in order to explain what they are and how to justify them. We describe good λ-frames $\mathfrak{s}$ in several stages. We need several choices to specify our context. Usually in model theory we fix an elementary class $\mathfrak{K}$ and consider $M \in \mathfrak{K}$. Here we concentrate on one cardinal λ, that is, we usually investigate $\mathfrak{K}_\lambda = (K_\lambda, \leq_{\mathfrak{K}_\lambda})$ where $K_\lambda = \{M \in K : M$ has cardinality $\lambda\}$ and $\leq_{\mathfrak{K}_\lambda}$ is defined by $M \leq_{\mathfrak{K}_\lambda} N$ iff $M \leq_{\mathfrak{K}} N, M \in K_\lambda$ and $N \in K_\lambda$. This is not a clear cut deviation, also for elementary classes we sometimes fix λ, and here we usually look at least at $\mathfrak{K}_\lambda$ and $\mathfrak{K}_{\lambda^+}$ together, still the flavour is different. So (the notion "choice" may be seemingly problematic but a better alternative was not found).

<u>2.22 Choice</u>: We concentrate on $\mathfrak{K}_\lambda$, an abstract elementary class restricted to one cardinal.

This seems reasonable because as noted above, transfer from one cardinal to another is central, but in our context quite hard, so we may know various "good" properties only around λ. Also there are $\mathfrak{K}$ which in some cardinals are model theoretically "very simple" but in other (e.g. larger) cardinals complicated, and we may like to say what we can say about $\mathfrak{K}_\lambda$ in λ for which $\mathfrak{K}_\lambda$ is "simple".

<u>2.23 Choice</u>: We concentrate here on $\mathfrak{K}_\lambda$ with amalgamation and the JEP (joint embedding properties).

But is amalgamation not a very strong/positive property? Yes, <u>but</u> amalgamation for models of cardinality λ only is much weaker and its failure in some reasonable circumstances leads to non-structure results, so it can serve as a dividing line. More specifically, we know that if $\mathfrak{K}$ is categorical in $\lambda \geq \text{LST}(\mathfrak{K})$ and $\mathfrak{K}_\lambda$ fails amalgamation and $\mathfrak{K}_{\lambda^+} \neq \emptyset$ then in $\mathfrak{K}_{\lambda^+}$ we have many complicated models (provided that $2^\lambda < 2^{\lambda^+}$; see Chapter I).

2.24 Choice: In $\mathfrak{K}_\lambda$ there is a <u>superlimit</u> model M^* which means that: $M^* \in \mathfrak{K}_\lambda$ is <u>universal</u>, (i.e., any $M' \in \mathfrak{K}_\lambda$ can be $\leq_\mathfrak{K}$-embedded into it), has a proper $<_\mathfrak{K}$-extension and if M is the union of a $<_\mathfrak{K}$-increasing chain of models isomorphic to M^* and M is of cardinality λ, then M is isomorphic to M^*.

Can we give a natural example of a superlimit model? For the abstract elementary class of linear orders, the rational order $(\mathbb{Q}, <)$ is superlimit (in $\aleph_0$). However, this is somewhat misleading as in larger cardinals it is much "harder", in fact, for the abstract elementary class of linear orders there is no superlimit model in $\lambda > \aleph_0$. By categoricity the abstract elementary class of algebraically closed fields of some fixed character has a superlimit model in every $\lambda \geq \aleph_0$. The class of $\{(A, E) : E$ an equivalence relation on $A\}$ is a bit more informative. Easily (A, E) is superlimit in it iff the number of E-equivalence classes as well as the cardinality of each E-equivalence class is the number of elements of A.

Of course, if $\mathfrak{K}$ is categorical in λ then every $M \in \mathfrak{K}_\lambda$ is superlimit (if it is not $\leq_\mathfrak{K}$-maximal in which case every $M \in \mathfrak{K}$ has cardinality $\leq \lambda$), but having a superlimit is a much weaker condition and it seems a right notion of generalizing superstability (or, probably, a good first approximation). This may surely look tautological in view of Definition 2.4(3), but that definition is misleading. There are several properties, which for elementary classes are equivalent to being superstable, and we have chosen the existence of superlimit. However, so far the existence of a superlimit model in λ has few consequences.

Why the choice? As this is an exterior way to say that our class is "simple, low"; it is weaker than categoricity and we next demand much more.

> Note that if $\mathfrak{K}$ is an elementary class and $\lambda = \lambda^{\aleph_0} + |\tau_\mathfrak{K}|$ or $\lambda \geq \beth_\omega + |\tau_\mathfrak{K}|$, <u>then</u> $M \in K_\lambda$, M is superlimit iff M is saturated and the theory is superstable; see [Sh 868, 3.1].

Now we are very interested in the existence of something like "free amalgamation", which in our context will be called non-forking amalgamation. That is, we are interested in saying when "M_1, M_2 are freely amalgamated over M_0 inside M_3" (all in $\mathfrak{K}_\lambda$). In our main example we have to use a more restrictive notion, having quadruples (M_0, M_1, a, M_3) is non-forking where $M_0 \leq_\mathfrak{K} M_1 \leq_\mathfrak{K} M_3, a \in M_3 \backslash M_1$. This says that "inside M_3 the element a and the model M_1 are freely amalgamated over M_0". (Mainly in [Sh 576], i.e. [Sh:E46], we use so called "minimal types", which give rise to such quadruples).

This leads us to define a central notion here: $\mathbf{tp}_\mathfrak{K}(a, M, N)$ denotes the "orbit" of $a \in N$ over $M \leq_\mathfrak{K} N$. We express (M_0, M_1, a, M_3) is non-forking also as "$\bigcup(M_0, M_1, a, M_3)$" and also as "$\mathbf{tp}_\mathfrak{s}(a, M_1, M_3)$ does not fork over M_0" because it is analogous to the non-forking in first order model theory. But this background is not needed, as non-forking is an abstract, axiomatic relation in our context.

This replaces here the notion of type in the investigation of elementary (= first order) classes. But there the types are defined as $\text{tp}(\bar{a}, A, N) = \{\varphi(\bar{x}, \bar{b}) : \bar{b} \subseteq A, \varphi(\bar{x}, \bar{y})$ is a first order formula and $N \models \varphi[\bar{a}, \bar{b}]\}$. Note: the case A is the universe of $M \leq_{\mathfrak{K}} N$ is not excluded but is not particularly distinguished. In fact, it was unnatural there to make the restriction as there are theorems using our ability to restrict the type to any subset of A (e.g. for inductive proof) and it is important to have results on any A.

We let $\mathscr{S}_{\mathfrak{K}_\lambda}(M) = \{\mathbf{tp}_{\mathfrak{K}_\lambda}(a, M, N) : M \leq_{\mathfrak{K}_\lambda} N$ and $a \in N\}$ be called the set of types over M. The set of axioms (i.e., Definition II.?) of good λ-frames expresses the intuition of "non-forking" as a free amalgamation (in fact we are allowed to restrict the non-forking relation to types $\mathbf{tp}_{\mathfrak{s}}(a, M_1, M_3)$ which are, so called basic ones, they should mainly be "dense" enough). We may consider these axioms per se, but we feel obliged to find evidence of their naturality of the form indicated above. So

2.25 Definition. A good λ-frame $\mathfrak{s}$ consists of

(a) an abstract elementary class $\mathfrak{K} = \mathfrak{K}^{\mathfrak{s}}$ and let[9] $\mathfrak{K}_{\mathfrak{s}} = \mathfrak{K}_\lambda$ with $\text{LST}(\mathfrak{K}^s) \leq \lambda$

(b) for $M \in \mathfrak{K}_\lambda$ we have $\mathscr{S}_{\mathfrak{s}}^{\text{bs}}(M)$, a subset of $\mathscr{S}_{\mathfrak{K}_\lambda}(M)$ with $\text{LST}(K^{\mathfrak{s}}) \leq \lambda$

(c) a notion of "$p \in \mathscr{S}^{\text{bs}}(M_2)$ does not fork over $M_1 \leq_{\mathfrak{K}_\lambda} M_2$" satisfying some reasonable axioms.

How does this help us in proving Theorem 2.20? Relying on the main results of [Sh 576], [Sh:E46], using the assumption of 2.20 we in II§3 prove that there is a good λ^+-frame $\mathfrak{s}$ with $\mathfrak{K}_{\mathfrak{s}} = \mathfrak{K}_{\lambda^+}$. Also in II§3 using a similar theorem from Chapter I for the case $\lambda = \aleph_0$ with a little different assumptions, we get a good $\aleph_0$-frame $\mathfrak{K}$.

We take a spiral approach: we look at a good λ-frame $\mathfrak{s}$, suggest a question, i.e., dividing lines, if $\mathfrak{s}$ falls under the complicated side we prove a non-structure theorem. If not, we know some things about it and we can continue to investigate it, after we have enough knowledge we ask another question. In II§5 we start with a good λ-frame, gain some knowledge and if there are not enough essentially unique amalgamations we get many complicated models in λ^{++}. If $\mathfrak{s}$ avoids this, we call it weakly successful and understand $\mathfrak{K}_{\mathfrak{s}}$ better. In particular, we define the promised "M_1, M_2 are non-forking amalgamated over M_0 inside M_3", we call this relation $\text{NF} = \text{NF}_\lambda = \text{NF}_{\mathfrak{s}}$ and prove that it has the properties hoped for. Listing its desired properties, it is unique. But this has a price: we have to restrict $\mathfrak{K}_{\mathfrak{s}}$ to isomorphic copies of the superlimit models. After showing that if S has a second

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[9] Note $\mathfrak{K}^{\mathfrak{s}}$ may have models in many cardinals, whereas $\mathfrak{K}_{\mathfrak{s}}$ has models in only one cardinal

non-structure property there are again many models in λ^{++}, we are "justified" in assuming $\mathfrak{s}$ fails also this non-structure a property. We then succeed to find for λ^+ another good frame, $\mathfrak{s}^+$ such that $K_{\lambda^+}^{\mathfrak{s}^+} \subseteq K_{\lambda^+}^{\mathfrak{s}}$. Recall $K^{\mathfrak{s}}$ is the a.e.c. lifting $K_{\mathfrak{s}}$ and $K_{\lambda^+}^{\mathfrak{s}} = (K^{\mathfrak{s}})_{\lambda^+}$.

What have we gained? Have we not worked hard just to find ourselves in the same place? Well, $\mathfrak{s}^+$ is a good λ^+-frame and $\dot{I}(\mu, K^{\mathfrak{s}^+}) \leq \dot{I}(\mu, K^{\mathfrak{s}})$ for every $\mu \geq \lambda^+$ and

   (∗) for every χ and good χ-frame $\mathfrak{t}, K^{\mathfrak{t}}$ has models of cardinality χ^+ and more-over of cardinality χ^{++}.

So this is enough to prove the Theorem 2.20(1), by induction on n.

Let us compare this to [Sh 87a], [Sh 87b]. There in stage n we have some knowledge on models in $\mathfrak{K}_{\aleph_\ell}$ for $\ell \leq n$ but our knowledge decreases with ℓ. Now (all in [Sh 87b]) dealing with $n + 1$ we have to consider a question on models of cardinality $\lambda = \aleph_0$, for which our specific tools for $\aleph_0$ (the omitting types theorem and the assumption that $\mathfrak{K}$ is $(\mathrm{Mod}_\psi, \prec)$ where $\psi \in \mathbb{L}_{\omega_1,\omega}$) enable us to have proved a dichotomy, each side implied additional information concerning $\aleph_\ell$ for $\ell \leq n$, again decreasing with ℓ.

> [We elaborate: for each $\ell < n$ we can define so called full stable $(\mathscr{P}^-(m), \aleph_\ell)$-systems $\langle M_u : u \in \mathscr{P}^-(m) \rangle$ for $m \leq (n - \ell)$ where $\mathscr{P}^-(m) = \{u : u \subset \{0, \ldots, m - 1\}\}$. So our knowledge "decreases" with ℓ: we can handle only systems of lower "dimension". We ask on such systems whether we can find suitable $M_{\{0,\ldots,n-1\}}$. Is it weakly unique (up to embedding)? Is it unique? Is there a prime one? We can transfer up a positive property from $(\mathscr{P}^-(m), \aleph_\ell)$ to $(\mathscr{P}^-(m - 1), \aleph_{\ell+1})$, and also negative ones if $2^{\aleph_\ell} < 2^{\aleph_{\ell+1}}$. A crucial point is the existence of a strong dichotomy in the cardinality $\aleph_0$, either we have a prime solution or we have $2^{\aleph_0}$ pairwise incompatible ones.
>
> Note that in [Sh 87a], [Sh 87b], we deal with types as in elementary classes (i.e. as set of formulas) but only over models or $\cup\{M_u : u \in \mathscr{P}^-(n)\}$ when $\langle M_u : u \in \mathscr{P}^-(n) \rangle$ is so called stable.]

The proofs of Chapter II seem neater than [Sh 87a], [Sh 87b]: because we are "poorer", we do not have the special knowledge on the first λ. So we do not have to look back, we can forget $\mathfrak{s}$ when advancing to $\mathfrak{s}^+$. This is nice for its purpose but suppose that we have a good λ^{+n}-frame $\mathfrak{s}^n$ for $n < \omega, \mathfrak{s}^{n+1}$ being gotten from $\mathfrak{s}^{+n}$ as above. For this purpose, forgetting the past costs us the future - we cannot say anything on models of cardinality $\geq \lambda^{+\omega}$. This is rectified in Chapter III.

So in Chapter III we investigate the $\mathfrak{K}_{\mathfrak{s}+n}$ for every n large enough.

A priori it is fine to do this for $n \geq 756$, and increasing the number as we continue to investigate. But in spite of this knowledge, considerable effort was wasted on small n, i.e., assuming little on $\mathfrak{s}$, and in III§2-§11 we get the theory of prime, independence, dimension, regular types and orthogonality we like (see, maybe, [Sh:F735] on what we really need to assume).

But for going up we need to deal with $\mathscr{P}^-(n)$-amalgamation - their existence and uniqueness. Then we can go up, see III§12.

§3 On Good λ-frames

This continues §2 and should be "non-logician friendly" too, though it may well be
more helpful after some understanding/reading of the material itself.

(A) Getting a good λ-frame

We try below to describe in more details the proof of Theorem 2.20(1) + (2) proved
in Chapter II, Chapter III, so we somewhat repeat what was said before in (D) of
§2. We have to start by getting good λ-frames. We could have concentrated on the
case $\lambda = \aleph_0$ and rely on Chapter I, but as this does not fit the "for non-logicians"
we instead rely on [Sh 576], [Sh 603], that is on [Sh:E46] and the non-structure
from [Sh 838], at least the "lean" version.

> For presentation we cheat a little in the non-structure part, saying
> we prove results like $\dot{I}(\mu^{++}, \mathfrak{K}) = 2^{\mu^{++}}$ when $\mathfrak{K}$ satisfies some "high"
> property and say $2^{\mu^+} < 2^{\mu^{++}}$. One point is that this relies on using
> an extra set theoretic assumption on μ^+: the weak diamond ideal on
> μ^+ not being μ^{++}-saturated. This is a very weak assumption, it is
> not clear whether its failure is consistent when $\mu \geq \aleph_1$ and in any case
> its failure has high consistency strength, that is, if the ideal is μ^{++}-
> saturated then there are inner models with quite large cardinals. We
> can eliminate this extra set theoretic assump[tion as done in [Sh 838]
> (see later part of the introduction). The second point is we prove only
> that there are $\geq \mu_{\text{unif}}(2^{\mu^{++}}, 2^{\mu^+})$ many non-isomorphic models in μ^{++}.
> This number is always $> 2^{\mu^+}$ (recall we are assuming $2^{\mu^+} < 2^{\mu^{++}}$),
> and is equal to $2^{\mu^{++}}$ when $\mu \geq \beth_\omega$ and conceivably the statement
> "$2^{\mu^+} < 2^{\mu^{++}} \Rightarrow \mu_{\text{unif}}(2^{\mu^{++}}, 2^{\mu^+}) = 2^{\mu^{++}}$" is provable in ZFC.
> Of course, below $\text{LST}(\mathfrak{K}) \leq \lambda$ suffices instead of $\text{LST}(\mathfrak{K}) = \aleph_0$.

So first assume

> $\boxdot_1$ $\mathfrak{K}$ is an abstract elementary class, and for simplicity $2^\lambda < 2^{\lambda^+} < \ldots <$
> $2^{\lambda^{+n}} < 2^{\lambda^{+n+1}} < \ldots, \mathfrak{K}$ is categorical in λ, λ^+, has a model in λ^{++}, and
> $\dot{I}(\lambda^{+2}, \mathfrak{K}) < 2^{\lambda^{+2}}$.

We shall now describe how to get a good λ-frame (or λ^+-frame) from this assume,
but it takes some time. We can deduce that $\mathfrak{K}_\lambda$ and $\mathfrak{K}_{\lambda^+}$ have amalgamation.
(Why? Otherwise it has many complicated models in λ^+, λ^{++}, respectively). Now
we consider the class $K_\lambda^{3,\text{na}}$ of triples $(M, N, a), M \leq_{\mathfrak{K}_\lambda} N, a \in N \backslash M$ with the
(natural) order which is $(M_1, N_1, a_1) \leq (M_2, N_2, a_2)$ iff $a_1 = a_2$ (yes! equal) and
$M_1 \leq_{\mathfrak{K}_\lambda} M_2$ and $N_1 \leq_{\mathfrak{K}_\lambda} N_2$.

We may look at them as representing the "orbit (or type of) a over M inside $N, \mathbf{tp}_{\mathfrak{K}}(a, M, N)$", which is not defined by formulas but by mappings, (i.e. types are orbits over M) so if $M \leq_{\mathfrak{K}_\lambda} N_\ell$ and $a_\ell \in N_\ell \backslash M$ then $\mathbf{tp}_{\mathfrak{K}_\lambda}(a_1, M, N_1) = \mathbf{tp}_{\mathfrak{K}}(a_2, M, N_2)$ iff for some $\leq_{\mathfrak{K}_\lambda}$-extension N_3 of N_2 there is a $\leq_{\mathfrak{K}_\lambda}$-embedding h of N_1 into N_3 over M which maps a_1 to a_2, recalling $\mathfrak{K}_\lambda$ has amalgamation.

Why do we consider $K_\lambda^{3,\mathrm{na}} := \{(M, N, a) : M \leq_{\mathfrak{K}_\lambda} N, a \in N \backslash M\}$ instead of $\mathscr{S}_{\mathfrak{K}_\lambda}^{\mathrm{na}}(M) := \{\mathbf{tp}_{\mathfrak{K}_\lambda}(a, M, N) : (M, N, a) \in K_\lambda^{3,\mathrm{na}}\}$? (The types $\mathbf{tp}_{\mathfrak{K}_\lambda}(a, M, N)$ when $a \in M$ are called algebraic (and na stands for non-algebraic) and are trivial, so $\mathscr{S}_{\mathfrak{K}_\lambda}^{\mathrm{na}}(M)$ is the rest.) Now $\mathscr{S}_{\mathfrak{K}_\lambda}(M)$ is very important and for $M_1 \leq_{\mathfrak{K}_\lambda} M_2, p \in \mathscr{S}_{\mathfrak{K}_\lambda}(M_2)$ we can define its restriction to $M_1, p \upharpoonright M_1 \in \mathscr{S}_{\mathfrak{K}_\lambda}(M_1)$, with some natural properties, and this mapping is onto (= surjective) as $\mathfrak{K}_\lambda$ has the amalgamation property. But it is not clear that an increasing sequence of types of length $\delta < \lambda^+$ of types has a bound (when $\mathrm{cf}(\delta) > \aleph_0$); see Baldwin-Shelah [BlSh 862]. For $K_\lambda^{3,\mathrm{na}}$ this holds. That is, if the sequence $\langle (M_\alpha, N_\alpha, a_\alpha) : \alpha < \delta \rangle$ is increasing in $K_\lambda^{3,\mathrm{na}}$, so $\alpha < \delta \Rightarrow a_\alpha = a_0$, then it has a lub: the triple $(\cup\{M_\alpha : \alpha < \delta\}, \cup\{N_\alpha : \alpha < \delta\}, a_0)$.

Some types (and triples) are in some sense better understood: here the ones representing minimal types; where

(∗) $p \in \mathscr{S}_{\mathfrak{K}_\lambda}^{\mathrm{na}}(M)$ is minimal if for every $\leq_{\mathfrak{K}_\lambda}$-extension N of M the type p has at most one extension in $\mathscr{S}_{\mathfrak{K}_\lambda}^{\mathrm{na}}(N)$.

Note that p always has at least one extension in $\mathscr{S}_{\mathfrak{K}_\lambda}(N)$ by amalgamation and we can prove that p has at least one from $\mathscr{S}_{\mathfrak{K}_\lambda}^{\mathrm{na}}(N)$ in our context, and recall that we have discarded the algebraic types, i.e. those of $a \in M$.

It is too much to expect that every $p \in \mathscr{S}_{\mathfrak{K}_\lambda}^{\mathrm{na}}(M)$ is minimal, but what about

<u>3.1 Question</u>: Is the class of minimal types dense, i.e., for every $p_1 \in \mathscr{S}_{\mathfrak{K}_\lambda}^{\mathrm{na}}(M_1)$ are there $M_2 \in \mathfrak{K}_\lambda$ and a minimal $p_2 \in \mathscr{S}_{\mathfrak{K}_\lambda}^{\mathrm{na}}(M_2)$ such that $M_1 \leq_{\mathfrak{K}_\lambda} M_2$ and p_2 extends p_1?

As we are assuming categoricity in λ and λ^+, this is not unreasonable and its failure implies having large $\mathscr{S}_{\mathfrak{K}_\lambda}^{\mathrm{na}}(M)$. Now [Sh:E46, §3,§4] relying on [Sh 838] (earlier: [Sh 603] and part of [Sh 576]) are dedicated to proving that the minimal 1-types are dense. (This requires looking more into the set theoretic side but also the model theoretic one; an example of a property which we consider is: given $M_0 <_{\mathfrak{K}_\lambda} M_1$ is there $M_2, M_0 <_{\mathfrak{K}_\lambda} M_2$ such that M_1, M_2 can be amalgamated over M_0 uniquely?).

So we assume the answer to 3.1 is yes; that is, we make the hypothesis:

3.2 Hypothesis. The answer to question 3.1 is yes: the minimal types are dense.

Having arrived here, further investigation shows

(∗) $\mathscr{S}_{\mathfrak{K}_\lambda}^{\mathrm{na}}(M)$ has cardinality $\leq \lambda$.

Now it is natural to define (M_0, M_1, a, M_3) is a non-forking quadruple or $\bigcup_{\mathfrak{s}}(M_0, M_1, a, M_3)$ iff $M_0 \leq_{\mathfrak{K}_\lambda} M_1 \leq_{\mathfrak{K}_\lambda} M_3, a \in M_3 \backslash M_1$ and $\mathbf{tp}_{\mathfrak{K}_\lambda}(a, M_0, M_3)$ is minimal. Recalling Candide we note that having chosen the unique non-trivial extension, we certainly have made the free choice: we have no freedom left on what is $\mathbf{tp}_{\mathfrak{K}_\lambda}(a, M_1, M_3)$! Now we find a good λ-frame $\mathfrak{s}$, with $\mathfrak{K}_{\mathfrak{s}} = \mathfrak{K}_\lambda$ and $\mathfrak{K}^{\mathfrak{s}} = \mathfrak{K}[\mathfrak{s}]$ will denote $\mathfrak{K}_{\geq \lambda} = \mathfrak{K} \upharpoonright \{M \in K : \|M\| \geq \lambda\}$ and the set of basic types, is $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{K}_\lambda}(M)$, the set of minimal $p \in \mathscr{S}^{\mathrm{na}}_{\mathfrak{K}_\lambda}(M)$. Note that good λ-frame is defined in II§2, existence in our case is proved in II§3.

> More accurately, in II§3 we prove in our present context the existence of a good λ^+-frame $\mathfrak{s}$ with $\mathfrak{K}_{\mathfrak{s}} = \mathfrak{K}_{\lambda^+}$, and we rely on having developed NF_λ in [Sh 576, §8]. But something parallel to [Sh 576, §8] is done in II§6 and described below. Moreover, in [Sh:E46] this is circumvented at the price of arriving to almost good λ-frame and then by [Sh 838] it is even a good λ-frame and it converges with the description here.

We assume here that $\mathfrak{K}_{\mathfrak{s}}(= \mathfrak{K}^{\mathfrak{s}}_\lambda)$ is categorical; in the present context this is reasonable (e.g., as otherwise you restrict yourself to $\{M \in \mathfrak{K}_{\mathfrak{s}} : M$ is superlimit$\}$).

(B) The successor of a good λ-frame

Now we look at our good λ-frame $\mathfrak{s}$, and the $\mathfrak{s}$-basic types in this case are the minimal types. But we can forget the minimality and just use the properties required in the definition of a good λ-frame (i.e. we are in Chapter II). Now as $M \in \mathfrak{K}_{\mathfrak{s}} \Rightarrow \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ has cardinality $\leq \lambda$ (by the definition of a good λ-frame) we can find $\leq_{\mathfrak{s}}$-increasing chains $\langle M_i : i \leq \lambda \times \delta \rangle$ such that for every $i < \lambda \times \delta$ every $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_i)$ is realized in M_{i+1}. In such a case we say that $M_{\lambda \times \delta}$ is brimmed over M_0. It follows that $M_{\lambda \times \delta}$ is determined uniquely up to isomorphisms over M_0 (seemingly, depending on $\mathrm{cf}(\delta) := \mathrm{Min}\{\mathrm{otp}(C) : C \subseteq \delta$ unbounded$\}$). Eventually we succeed to prove that the choice of the limit ordinal $\delta(< \lambda^+)$ is immaterial, see II§1,§4.

> (These are relatives of universal homogeneous, saturated models and special models.)

We define $K^{3,\mathrm{bs}}_{\mathfrak{s}}$ as the class of triples (M, N, a) such that $M \leq_{\mathfrak{K}_{\mathfrak{s}}} N$ and $\mathbf{tp}_{\mathfrak{K}_{\mathfrak{s}}}(a, M, N) \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$. By the axioms of "good λ-frames" for $(M_1, N_1, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ and M_2 such that $M_1 \leq_{\mathfrak{s}} M_2$ we can find $M'_2 \in \mathfrak{K}_\lambda$ isomorphic to M_2 over M_1 and $N_2 \in \mathfrak{K}_\lambda$, which is $\leq_{\mathfrak{K}}$-above M'_2 and N_1 and $\mathbf{tp}_{\mathfrak{s}}(a, M'_2, N_2)$ does not fork over M_1. In this case we say $(M_1, N_1, a) \leq_{\mathfrak{s}} (M'_2, N_2, a)$, (or use $\leq_{\mathrm{bs}} = \leq^{\mathfrak{s}}_{\mathrm{bs}}$ instead $\leq_{\mathfrak{s}}$).

Having existence is nice, but having also uniqueness is better. So we become interested in $K^{3,\mathrm{uq}}_{\mathfrak{s}}$, the class of $(M, N, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ satisfying: if $(M_*, N_*, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$

is $\leq_{\mathfrak{s}}$-above (M, N, a), then the way M_*, N are amalgamated over M inside N_* is <u>unique</u> (up to common embeddings).

> For the first order case this means "tp$(N, M \cup \{a\})$ is weakly orthogonal to M"; (i.e., domination).

<u>3.3 Question</u>: 1) (Density) Do we have "$K_{\mathfrak{s}}^{3,\mathrm{uq}}$ is dense in $K_{\mathfrak{s}}^{3,\mathrm{bs}}$ (under $\leq_{\mathfrak{s}}$)"?
2) (Existence) Assume $p \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M)$, can we find a, N such that $(M, N, a) \in K_{\mathfrak{s}}^{3,\mathrm{uq}}$ and $\mathbf{tp}_{\mathfrak{s}}(M, N, a) = p$?

As $\mathfrak{K}_{\mathfrak{s}}$ is categorical, we can prove that density implies existence.

"Have we not been here before?" the reader may wonder. This is the spiral phenomena: in 3.1 we were interested in a different kind of uniqueness. Now we prove that the non-density is a non-structure property and as a token of our pleasure, $\mathfrak{s}$ with positive answer is called weakly successful.

3.4 Hypothesis. The answer to 3.3 is yes, enough triples in $K_{\mathfrak{s}}^{3,\mathrm{uq}}$ exist.

So we have some cases of uniqueness of the non-forking amalgamation. When we (in II§6) close this family of cases of uniqueness, under transitivity and monotonicity we get a four-place relation $\mathrm{NF}_\lambda = \mathrm{NF}_{\mathfrak{s}}$ on $\mathfrak{K}_\lambda$. Working enough we show that $\mathrm{NF}_{\mathfrak{s}}$ conforms reasonably with "M_1, M_2 and are in non-forking ($\equiv$ free) amalgamation over M_0 inside M_3". We justify the definition showing that some natural properties it satisfies has at most one solution (for any good λ-frame).

Now we start to look at models in $K_{\lambda^+}^{\mathfrak{s}}$; in an attempt to find a good λ^+-frame $\mathfrak{s}^+ = s(+)$, a successor of $\mathfrak{s}$. There are some models in $K_{\lambda^+}^{\mathfrak{s}}$; in fact, there is a universal homogeneous one M^* and it is unique so if there is a superlimit $M \in K_{\lambda^+}^{\mathfrak{s}}$ then $M \cong M^*$. Now if $\langle M_i : i < \lambda^+ \rangle$ is $\leq_{\mathfrak{K}_{\lambda^+}^{\mathfrak{s}}}$-increasing $M_i \cong M^*$ then $\cup\{M_i : i < \lambda^+\} \cong M^*$ but it is not clear if, e.g., $\cup\{M_i : i < \omega\} \cong M^*$. So we consider another choice of being a substructure in $K_{\lambda^+}^{\mathfrak{s}} : M_1 \leq_{\lambda^+}^* M_2$ iff $M_1, M_2 \cong M^*$ and for some $\leq_{\mathfrak{K}}$-representations (also called $\leq_{\mathfrak{K}}$-filtrations) $\langle M_\alpha^\ell : \alpha < \lambda^+ \rangle$ of M_ℓ for $\ell = 1, 2$ we have $\mathrm{NF}_{\mathfrak{s}}(M_i^1, M_i^2, M_j^1, M_j^2)$ for every $i < j < \lambda^+$.

> [We say that $\langle M_\alpha : \alpha < \lambda^+ \rangle$ is a $\leq_{\mathfrak{K}}$-representation or $\leq_{\mathfrak{K}}$-filtration of $M \in \mathfrak{K}_{\lambda^+}$ when $M_\alpha \in \mathfrak{K}_\lambda$ is $\leq_{\mathfrak{K}_\lambda}$-increasing continuous for $\alpha < \lambda^+$ and $M = \cup\{M_\alpha : \alpha < \lambda^+\}$.]

We would love to understand $\mathfrak{K}_{\lambda^+}$, but this seems too hard, so presently so we restrict ourselves to isomorphic copies of the model we do understand, M^*.

> This conforms with the strategy of first understanding the quite saturated models.

This helps to prove "M^* is superlimit" but with a price: we have to consider the following question.

<u>3.5 Question</u>: Assume $\langle M_i : i \leq \delta \rangle$ is $\leq^*_{\lambda^+}$-increasing continuous, δ a limit ordinal $< \lambda^{++}$ and $i < \delta \Rightarrow M_i \cong M^*$ and $i < \delta \Rightarrow M_i \leq^*_{\lambda^+} N$ and $N \cong M^*$. Does it follow that $M_\delta \leq^*_{\lambda^+} N$?

This is an axiom of an abstract elementary class, so we know that it holds for $(\mathfrak{K}_{\lambda^+}, \leq_{\mathfrak{K}})$ but not necessarily for $\leq^*_{\lambda^+}$. This is another dividing line: if the answer is no, we get a non-structure theorem. If the answer is yes, we call $\mathfrak{s}$ successful.

3.6 Hypothesis. $\mathfrak{s}$ is successful.

We go on and prove that $\mathfrak{s}^+$ is a good λ^+-frame. Well, the reader may wonder: all this work and you just end up where you have started, just one cardinal up? True, but if $\mathfrak{s}$ is a good λ-frame then $K^{\mathfrak{s}}_{\lambda^{++}} \neq \emptyset$, so for a successful $\mathfrak{s}$, applying this to the good λ^+-frame $\mathfrak{s}^+$ we get $\mathfrak{K}_{\lambda_{\mathfrak{s}}^{+3}} \neq \emptyset$. Having "arrived to the same place one cardinal up" is enough to prove part (1) of Theorem 2.20!

More elaborately, under the assumptions of 2.20 there is a good λ^+-frame $\mathfrak{s}_1$ with $\mathfrak{K}^{\mathfrak{s}_1} \subseteq \mathfrak{K}^{\mathfrak{s}}$. Second, if we prove by induction on $k = 1, \ldots, n-1$ that there is a good λ^{+k}-frame $\mathfrak{s}_k$ with $K^{\mathfrak{s}_k} \subseteq K^{\mathfrak{s}_{k-1}}$, the induction step is what we have proved. For $k = n-1$, "$K^{\mathfrak{s}_k}$ has a model in $\lambda^{++}_{\mathfrak{s}_k}$" means that $K_{\lambda^{+n+1}} \neq \emptyset$ as asked for in 2.20(1). All this is Chapter II, so its proof proceeds by "forgetting" the previous $\mathfrak{s}$ when advancing $\mathfrak{s}^+$ and $\lambda^+_{\mathfrak{s}}$. Next assume

$\square_2$ $\mathfrak{s}$ is a λ-good frame, $\dot{I}(\lambda^{+n}, \mathfrak{K}^{\mathfrak{s}}) < 2^{\lambda^{+n}}$ and $2^{\lambda^{+n}} < 2^{\lambda^{+n+1}}$ for $n < \omega$.

We now define by induction on n a good λ^{+n}-frame $\mathfrak{s}^{+n} = \mathfrak{s}(+n)$. Let $\mathfrak{s}^0 = \mathfrak{s}$ and having defined $\mathfrak{s}^{+n}$, it has to be successful by the previous argument so $\mathfrak{s}^{+(n+1)} := (\mathfrak{s}^{+n})^+$ is a well defined good $\lambda^{+(n+1)}$-frame. We can prove by induction on n that $K_{\mathfrak{s}(+n)} \subseteq K^{\mathfrak{s}}$ and $m < n \Rightarrow K^{\mathfrak{s}(+n)} \subseteq K^{\mathfrak{s}(+m)}$.

Note that if $K^{\mathfrak{s}}$ is the class of (A, E) where $|A| \geq \lambda$ and E is an equivalence relation on A then $K^{\mathfrak{s}^{+n}}$ is the class of $(A, E) \in K^{\mathfrak{s}}$ such that E has $\geq \lambda^{+n}$ equivalence classes each of cardinality $\geq \lambda^{+n}$.

(C) The beauty of ω successive good λ-frames

What about part (2) of 2.20, i.e., models in cardinalities $\geq \lambda^{+\omega}$? The connection between $\mathfrak{s}^{+n}, \mathfrak{s}^{+(n+1)}$ is not strict enough. Now though we have $K^{\mathfrak{s}^{+n+1}} \subseteq K^{\mathfrak{s}^{+n}}$, we do not know whether $\leq_{\mathfrak{s}(+n+1)}$ is $\leq_{\mathfrak{K}[\mathfrak{s}(+n)]} \restriction K_{\mathfrak{s}(+n+1)}$ and whether $\mathfrak{K}_{\mathfrak{s}(+n+1)} = K^{\mathfrak{s}(+n)}_{\lambda^{+n+1}}$. We can overcome the first problem. We show that if $\mathfrak{s}$ is so called good$^+$ then $\leq_{\mathfrak{s}(+)} = \leq_{\mathfrak{K}[\mathfrak{s}]} \restriction K_{\mathfrak{s}(+)}$ (and $\mathfrak{s}$ is good$^+$ "usually" holds e.g., if $\mathfrak{s} = \mathfrak{t}^+$, $\mathfrak{t}$ is good$^+$ and successful, see III§1). In this case $\langle \mathfrak{K}^{\mathfrak{s}^{+n}} : n < \omega \rangle$ is decreasing and even

$\langle K_{\lambda+n}^{\mathfrak{s}+m} : m \leq n \rangle$ is decreasing in m, but the orders agree when well defined. The crux of the matter is in the end (III§12, relying on what we prove earlier in Chapter III), to show that for some $\mathfrak{s}^{+\omega}$, $K_{\mathfrak{s}+\omega} = \cap\{\mathfrak{K}_{\lambda+\omega}^{\mathfrak{s}(+n)} : n < \omega\}$ and $\mathfrak{s}^{+\omega}$ is so called beautiful, so at last we shall arrive to "the promised land" from $\boxtimes(c)$ from the beginning of §2(C). But this comes only at the very end. In particular before starting we have to know much on the $\mathfrak{K}_{\mathfrak{s}(+n)}$'s. It is enough to prove any of the nice things we like to know on $\mathfrak{K}_{\mathfrak{s}(+n)}$ just for "$n < \omega$ large enough". A priori we may have from time to time to say "if $\mathfrak{s}$ has the desirable properties $(A)_1, \ldots, (A)_{\ell-1}$ then $\mathfrak{s}^{+n}$ has $(A)_\ell$ (as we are assuming all $\mathfrak{s}^{+n}(n < \omega)$ are successful), and so when we prove a desirable property X we prove it for $\mathfrak{s}^{+n}$ when $n \geq n_X$". Originally we were using $n \geq 2$ or $n \geq 3$, but try to use little, say "$\mathfrak{s}$ is weakly successful" (which means n is 0 or 1) and lately try just to finish.

Note also that without loss of generality $\mathfrak{s}$ is type-full, i.e. $\mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M) = \mathscr{S}_{\mathfrak{s}}^{\mathrm{na}}(M)$, as we can use our knowledge on $\mathrm{NF}_{\mathfrak{s}}$ to define when "$p \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{na}}(N)$ does not fork over $M \leq_{\mathfrak{s}} N$" and prove that $\mathfrak{t}$ is a good λ-frame when we define $\mathfrak{t}$ by $\mathfrak{K}_{\mathfrak{t}} = \mathfrak{K}_{\mathfrak{s}}, \mathscr{S}_{\mathfrak{t}}^{\mathrm{bs}} = \mathscr{S}_{\mathfrak{s}}^{\mathrm{na}}$, and non-forking as above. As we can replace $\mathfrak{s}$ by $\mathfrak{t}$ the "w.l.o.g." above is justified.

Note that the $\mathfrak{K}_{\mathfrak{s}(+n)}$ are categorical, but this is deceptive: $\mathfrak{K}_{\mathfrak{s}(+n)}$ is, but $K_{\lambda+n+1}^{\mathfrak{s}(+n)}$ is not necessarily categorical. So in order to eventually understand the categoricity spectrum in III§2 we sort out when is $\mathfrak{K}_{\lambda+}^{\mathfrak{s}}$ categorical (for a successful good λ-frame $\mathfrak{s}$).

We define several (variants of) $\mathfrak{s}$ is uni-dimensional, prove the equivalence with "$K^{\mathfrak{s}}$ is categorical in $\lambda_{\mathfrak{s}}^+$" and show that (for successful $\mathfrak{s}$) $\mathfrak{s}$ is uni-dimensional iff $\mathfrak{s}^+$ is uni-dimensional (so this applies to $\mathfrak{s}^{+n}$ and $\mathfrak{s}^{+(n+1)}$ when well defined). So in the case we have chosen, $\mathfrak{s}^+, \mathfrak{s}^{+2}, \ldots$ are uni-dimensional and $K_{\lambda+n}^{\mathfrak{s}(+n)} = K_{\lambda+n}^{\mathfrak{s}}$ so in the beautiful (see below) case it implies categoricity in all $\mu > \lambda$.

We now review Chapter III in more detail. We define and investigate "$\mathbf{J}$ is a set of elements in $N \backslash M$ which is independent over M" in symbols $(M, N, \mathbf{J}) \in K_{\mathfrak{s}}^{3,\mathrm{bs}}$. The idea is that if $\langle M_i : i \leq \alpha \rangle$ is $\leq_{\mathfrak{s}}$-increasing, $a_i \in M_{i+1} \backslash M_i$ and $\mathbf{tp}_{\mathfrak{s}}(a_i, M_i, M_{i+1})$ does not fork over M_i for $i < \alpha$, then $(M_0, M_\alpha, \{a_i : i < \alpha\}) \in K_{\mathfrak{s}}^{3,\mathrm{bs}}$ and even $(M_0, M', \{a_i : i < \alpha\}) \in K_{\mathfrak{s}}^{3,\mathrm{bs}}$ if $M \cup \{a_i : i < \alpha\} \subseteq M' \leq_{\mathfrak{s}} M_\alpha$. But we have to prove that this notion has the expected properties, e.g., the finite character (see III§5).

We know about $(M, N, a) \in K_{\mathfrak{s}}^{3,\mathrm{uq}}$, but also important is $(M, N, a) \in K_{\mathfrak{s}}^{3,\mathrm{pr}}$: the triple is prime, i.e., such that if $(M, N', a') \in K_{\mathfrak{s}}^{3,\mathrm{bs}}$ and $\mathbf{tp}_{\mathfrak{s}}(a, M, N) = \mathbf{tp}_{\mathfrak{s}}(a', M, N')$ then there is a $\leq_{\mathfrak{s}}$-embedding of N into N' over M mapping a to a'. We prove existence in enough cases (mainly for $\mathfrak{s}^+$) and eventually define and investigate also "N is prime over $M \cup \mathbf{J}$" when $(M, N, \mathbf{J}) \in K_{\mathfrak{s}}^{3,\mathrm{bs}}$ and $\mathbf{J}$ is maximal.

Next we develop orthogonality: assume $p_\ell \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M)$ for $\ell = 1, 2$. Then $p_1 \perp p_2$

when: if $(M, N, a) \in K_{\mathfrak{s}}^{3,\mathrm{uq}}$ and $p_1 = \mathbf{tp}_{\mathfrak{s}}(a, M, N)$ then p_2 has a unique extension in $\mathscr{S}_{\mathfrak{s}}(N)$. This means that there is no connection, no interaction between p_1 and p_2. It implies that $(M, N, \{a_i : i < \alpha\}) \in K_{\mathfrak{s}}^{3,\mathrm{bs}}$, i.e., is independent iff for each $j < \alpha, (M, N, \{a_i : i < \alpha, p_j \perp p_i\})$ is independent where $p_i = \mathbf{tp}_{\mathfrak{s}}(a_i, M, N)$. We prove that this behaves reasonably; in particular, is preserved by non-forking extensions. We similarly define $p \perp M$ (when $M \leq_{\mathfrak{s}} N, p \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(N)$). Because of the categoricity (and $\mathfrak{s} = \mathfrak{t}^+$) we can prove $K_{\mathfrak{s}}^{3,\mathrm{pr}} = K_{\mathfrak{s}}^{3,\mathrm{uq}}$.

In those terms we can characterize when $(M, N, a) \in K_{\mathfrak{s}}^{3,\mathrm{bs}}$ has uniqueness (i.e., $\in K_{\mathfrak{s}}^{3,\mathrm{uq}}$), under the assumption that there are primes. It holds <u>iff</u> there is a decomposition $\langle (M_i, a_j) : i \leq \alpha, j < \alpha \rangle$ of (M, N), i.e., $M_0 = M, M_\alpha = N, (M_i, M_{i+1}, a_i) \in K_{\mathfrak{s}}^{3,\mathrm{pr}}$ such that $a_0 = a$ and $i \in (0, \alpha) \Rightarrow \mathbf{tp}_{\mathfrak{s}}(a_i, M_i, M_{i+1}) \perp M_0$. We can define regular types such that: for $M \leq_{\mathfrak{s}} N$ and regular $p \in \mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M)$ the dependence relation on $\mathbf{I}_{M,N} = \{a \in N : a \text{ realizes } p\}$ behaves as independence in vector spaces (for others it behaves like sets of finite sequences from a vector space), and regular types are dense (i.e., if $M <_{\mathfrak{s}} N$ then for some $a \in N \backslash M, \mathbf{tp}_{\mathfrak{s}}(a, M, N)$ is regular). So $a \in \mathbf{I}_{M,N}$ depends on $\mathbf{J} \subseteq \mathbf{I}_{M,N}$ iff there are $M_1 \leq_{\mathfrak{K}_{\mathfrak{s}}} N_1$ such that $M \leq_{\mathfrak{K}_{\mathfrak{s}}} M_1, N \leq_{\mathfrak{K}_{\mathfrak{s}}} N_1, \mathbf{J} \subseteq M_1$, the triple $(M, N, \mathbf{J})$ has uniqueness and $\mathbf{tp}_{\mathfrak{s}}(a, M_1, N_1)$ forks over M. It has local character (if $a \in \mathbf{I}_{M,N}$ depends on $\mathbf{J}$ then it depends on some finite subsets of it), monotonicity, transitivity (if $a \in \mathbf{I}_{M,N}$ depends on $\mathbf{J}' \subseteq \mathbf{I}_{M,N}$ and each $b \in \mathbf{J}'$ depends on $\mathbf{J} \subseteq \mathbf{I}_{M,N}$ then a depends on $\mathbf{J}$) and satisfies the exchange lemma. Then we can define (and prove the relevant properties) when "$\{M_i : i < \alpha\}$ is independent over M inside N" and we can deal similarly with "$\langle M_\eta : \eta \in \mathscr{T} \rangle$ is independent inside N" when $\mathscr{T} \subseteq {}^{\omega >}(\lambda_{\mathfrak{s}})$ is closed under initial segments.

We may now consider the main gap in this context (but mostly this is delayed). From some perspective this is ridiculous: $\mathfrak{K}_{\mathfrak{s}}$ is categorical in $\lambda_{\mathfrak{s}}$. But we analyze $\{N : M_* \leq_{\mathfrak{s}} N\}$ for a fixed M_*, so all the models in this class have cardinality λ_s. (In this still there is some degeneration, but we can analyze models from $\mathfrak{K}_{\lambda^+}^{\mathfrak{s}}$, in this case there is no real difference between what we do and the actual main gap theorem, so again all models have a fixed cardinality. And if $\mathfrak{s}$ is beautiful, see below, we can do the same for $\mathfrak{K}^{\mathfrak{s}}$. If $\mathfrak{s}$ is "good enough up to λ^{+n} we can deal similarly with $K_{\leq \lambda^{+n}}^{\mathfrak{s}}$).

So if $M \leq_{\mathfrak{s}} N$ (assuming, e.g. $\mathfrak{s}$ is a successful λ-frame with primes; less is needed), we can find a decomposition $\langle N_\eta, a_\nu : \eta \in \mathscr{T}, \nu \in \mathscr{T} \backslash \{<>\} \rangle$ of N which means

⊛ (a) $\mathscr{T} \subseteq {}^{\omega >}(\lambda_{\mathfrak{s}})$ is non-empty closed under initial segments

(b) $N_\eta \leq_{\mathfrak{s}} N$

(c) $\nu \triangleleft \eta \Rightarrow N_\nu \leq_{\mathfrak{s}} N_\eta$

(d) $(N_\eta, N_{\eta\hat{}<\alpha>}, a_{\eta\hat{}<\alpha>}) \in K_{\mathfrak{s}}^{3,\mathrm{pr}}$ if $\eta\hat{} < \alpha > \in \mathscr{T}$

(e) $\{a_{\eta^\frown<\alpha>} : \eta^\frown < \alpha > \in \mathscr{T}\}$ is independent in (M_η, N) and is
a maximal such set (with no repetitions, of course)

(f) $N_{<>} = M$,

(g) if $\cup\{N_\eta : \eta \in \mathscr{T}\} \subseteq N' <_\mathfrak{s} N, p = \mathbf{tp}_\mathfrak{s}(a, N', N) \in \mathscr{S}^{\mathrm{bs}}_\mathfrak{s}(N')$
then $p \pm N_\eta$ for some $\eta \in \mathscr{T}$

(h) if $\nu \lhd \eta \lhd \eta^\frown\langle\alpha\rangle \in \mathscr{T}$ then $\mathbf{tp}_\mathfrak{s}(a_{\eta^\frown\langle\alpha\rangle}, N_\eta, N_{\eta^\frown\langle\alpha\rangle}) \perp N_\nu$.

3.7 Question: Is always N prime and/or minimal over $\cup\{N_\eta : \eta \in \mathscr{T}\}$?

The answer is yes iff whenever $\mathscr{T} = \{<>, < 0 >, < 1 >\}$ the answer is yes and we then say that $\mathfrak{s}$ have the so-called NDOP. Moreover, its negation DOP is a strong non-structure property: for every $R \subseteq \lambda \times \lambda$ we can find $N_R \in K^\mathfrak{s}_{\lambda^{++}}$ and $\bar{a}_\alpha, \bar{b}_\alpha \in {}^{\lambda_\mathfrak{s}}(N_R)$ for $\alpha < \lambda$ such that some condition (preserved by isomorphism) is satisfied by $\bar{a}_\alpha{}^\frown\bar{b}_\beta$ in N_R iff $(\alpha, \beta) \in R$. Also the NDOP holds for $\mathfrak{s}^+$ iff it holds for $\mathfrak{s}$ when $\mathfrak{s}$ is successful from DOP. We can get $\dot{I}(\lambda^{++}_\mathfrak{s}, K^\mathfrak{s}) = 2^{\lambda^{++}_\mathfrak{s}}$ and more.

$$* \qquad * \qquad *$$

How does all this help us to go up? That is, we assume $\mathfrak{s}^{+n}$ is well defined and successful for every n (equivalently $\mathfrak{s}$ is n-successful for every n) and we would like to understand the models in $\mathfrak{K}^{\mathfrak{s}(+\omega)}$, (so they have cardinality $\geq \lambda^{+\omega}$ and are close to being $\lambda^{+\omega}$-saturated). The going up is done in the framework of stable $\mathscr{P}^{(-)}(n)$-system of models $\langle M_u : u \in \mathscr{P}^-(n)\rangle$, $\mathscr{P}^-(n) = \{u : u \subset \{0, \ldots, n-1\}$; explained below. This is done in III§12 (which should be helpful for completing [Sh 322]).

In short, to understand existence/uniqueness of models (and of amalgamation) in λ, we consider such properties for some n-dimensional systems of models in every large enough $\mu \leq \lambda$. So for $n = 0, 1, 2$ we get the original problems but understanding the n-th case given in λ is intimately connected to understand the $(n+1)$-case for every large enough $\mu < \lambda$. So for $\lambda = \mu^+$ we get a positive property for (μ^+, n) from one for $(\mu, n+1)$.

Why do we need such systems? Consider $\lambda_* \geq \mu_* \geq \lambda_\mathfrak{s}$ and we try to analyze models of cardinality $\in [\mu_*, \lambda_*]$ by pieces of cardinality μ_* or $\mu' \in [\mu_*, \lambda_*)$ (in the end we consider $\mu_* = \lambda^{+\omega}_\mathfrak{s}$, but most of the analysis is for the case $\lambda_*, \mu_* \in [\lambda_\mathfrak{s}, \lambda^{+\omega}_\mathfrak{s})$). We can analyze a model M from $\mathfrak{K}$ of cardinality $\lambda_0 \in (\mu_*, \lambda_*]$ by a $\leq_\mathfrak{K}$-increasing continuous sequence $\langle M_\alpha : \alpha < \lambda_0 \rangle, \mu_* \leq \|M_\alpha\| = \|M_{\alpha+1}\| < \lambda_0$, with $M = \cup\{M_\alpha : \alpha < \lambda_0\}$; so it suffices to analyze $M_{\alpha+1}$ over M_α for each α. We can analyze M_1 over M_0 for a pair of models $M_0 \leq_\mathfrak{K} M_1$ of the same cardinality which we call λ_1 when $\lambda_1 > \mu_*$ by an $(\leq_\mathfrak{K})$-increasing continuous sequence of pairs $\langle (M^0_i, M^1_i) : i < \lambda_1 \rangle$ where $\|M^0_i\| = \|M^0_{i+1}\| = \|M^1_i\| = \|M^1_{i+1}\| < \lambda_1$, and we have to analyze M^1_{i+1} over $\langle M^0_i, M^1_i, M^0_{i+1}\rangle$ for each i. In the next stage we have $8 = 2^3$

models and have to analyze the largest over the rest. Eventually we arrive to the case that all of them have cardinality μ_*.

In short, we have to consider suitable $\mathscr{P}(n)$-systems $\langle M_u : u \in \mathscr{P}(n) \rangle$ where $\mathscr{P}(n) = \{u : u \subseteq \{0, \ldots, n-1\}\}, u \subseteq v \Rightarrow M_u \leq_{\mathfrak{K}^{\mathfrak{s}}} M_v$ and $\|M_u\| = \|M_0\| \in [\mu_*, \lambda_*]$.

We would like to analyze $M_{\{0,\ldots,n-1\}}$ over $\cup\{M_u : u \in \mathscr{P}^-(n)\}$ where $\mathscr{P}^-(n) = \mathscr{P}(n)\backslash\{0, \ldots, n-1\}$. Such analysis of a "big" system of small models naturally help proving cases of uniqueness, e.g., uniqueness of non-forking-amalgamations suitably defined. So if for μ_* we have positive answers for every n, then this holds for every $\lambda \in [\mu_*, \lambda_*]$.

But we are interested as well in existence proofs. (Note that in the proof we have to deal with uniqueness, existence (and some relatives) simultaneously.) For the existence we need for a given suitable system $\langle M_u : u \in \mathscr{P}^-(n) \rangle$ to complete it by finding $M_{\{0,\ldots,n-1\}}$. Well, but what are the suitable systems? Those are defined, by several demands including $u \subseteq v \Rightarrow M_u \leq_{\mathfrak{K}^{\mathfrak{s}}} M_v$ (and many more restrictions which hold if the sequence of approximations chosen above are "fast" enough). We called them the stable ones. For each n, k we can ask on $\mathfrak{s}^{+n}$ some questions on $\mathscr{P}(k)$-systems: mainly versions of existence and uniqueness. A major point is that failure of uniqueness for $\lambda^{+n}, \mathscr{P}(m+1)$ implies failure for $\lambda^{+n+1}, \mathscr{P}(m)$ (using $2^{\lambda^{+n}} < 2^{\lambda^{+n+1}}$). But to get strong dichotomy we have to use systems which have the right amount of brimmness. At last we have a glimpse of "paradise", we can define when $\mathfrak{s}$ is n-beautiful essentially when it satisfies all the good properties on stable $\mathscr{P}(m)$-systems for $m \leq n$. In the end we prove that $\mathfrak{s}^{+n}$ is $(n+2)$-beautiful, i.e. has all the desired properties for $m \leq n+2$ but for this we use $\mathfrak{s}^{n+\ell}$ being successful for $\ell \leq n$.

Having all this we can prove that $\mathfrak{s}^{+\omega}$ has all the good properties (but we have to work on changing the brimmness demands) so is ω-beautiful. This now can be lifted up, in particular $\mathfrak{K}^{\mathfrak{s}(+\omega)}$ has amalgamation and the types $\mathbf{tp}_{\mathfrak{K}[\mathfrak{s}(+\omega)]}(a, M, N)$ are μ-local for $\mu = \lambda^{+\omega}$ (in fact $\mu = \lambda$ is enough) where

(∗) $\mathfrak{K}$ an abstract elementary class with amalgamation, is μ-local when for $M \leq_{\mathfrak{K}} N$ and $a_1, a_2 \in N$ we have:
$\mathbf{tp}_{\mathfrak{K}}(a_1, M, N) = \mathbf{tp}_{\mathfrak{K}}(a_2, M, N)$ <u>iff</u> for every $M' \leq_{\mathfrak{K}} M$ of cardinality $\mu, \mathbf{tp}_{\mathfrak{K}}(a_2, M', N) = \mathbf{tp}_{\mathfrak{K}}(a_2, M', N)$.

Now for a beautiful $\mathfrak{s}$, in particular we have amalgamation/stable amalgamation, prime models over a triple of models in stable amalgamation. In particular we can prove the main gap. However, here we just present the characterization of the categoricity spectrum (see 2.20(2)) and delay the rest.

On Chapter IV and [Sh 838] see §4(B).

We may wonder how excellent classes and beautiful good λ-frames are related. We explain this by comparing each to the first order cases.

If T is a (first order complete) theory T which is $\aleph_0$-stable then $(\mathrm{Mod}_T, \prec)$ is an excellent class. Pedantically assume T categorical in $\aleph_0$. Better expand each $M \in \mathrm{Mod}_T$ to $M^+ = (M, P_P^{M^+})_{P \in D(T)}$ with $P_{p(x)}^{M^+} = \{\bar{a} \in {}^{\ell g(\bar{a})}M : \mathrm{tp}(\bar{a}, \emptyset, M) = p(x)\}$; as written in [Sh 82] categoriticity in $\aleph_0$, is assume but this is not used in any real way.

For an excellent class for simplicity is, for notational transparency, the class of almac models of a first order T. Now we continue to use "classical" types and (respecting atomicity) have primes even primary models but only over sets like $\cup\{M_u : u \in p/n\}$ inside M where $\bar{M} = \langle M_u : u \in \mathscr{P}(n) \rangle$ is a stable system of models. Stable non-forking is defined only for such $\bar{M}$'s but still using formulas and (classical) types. Big differences, but usually any concept that is not obviously irrelevant can be developed in this context.

If T is a superstable first order T stable in λ <u>then</u> there is a beautiful λ-frame $\mathfrak{s} = \mathfrak{s}_{T,\aleph_0}, K^{\mathfrak{s}}$ is a class of $\aleph_\varepsilon$-saturated models of T of cardinality $\geq \lambda, \leq_{\mathfrak{s}}$ is $\prec$ on this class. But for general beautiful types are defined without formulas - by orbits (unlike excellent classes). As in excellent classes stable or non-forking systems $\langle M_u : u \in \mathscr{P} \subseteq \mathscr{P}(u) \rangle$ are central, but their definition is not direct, certainly not referring to classical types. Also "M_1 is prime over M^+" is again central, but defines by arrow (no "primary model".

Again usually whatever we can prove for $\mathfrak{s}$ of the form $\varepsilon - \mathfrak{s}_{T,\aleph_\varepsilon}$, see above, and is not obviously irrelevant can be proved in this content.

§4 Appetite comes with eating

Here we mainly review open questions, Chapter IV, [Sh 838] and further relevant works which could have been part of this book but were not completely ready; so decided not to wait because my record of dragging almost finished books is bad enough even without this case. Note that Chapter IV use infinitary logics and most of [Sh 838] has largely set theoretic character hence does not fit with §2,§3.

But we begin by looking at what has been described so far has not accomplished. (By this division we end up dealing with some issues more than once.)

(A) <u>The empty half of the glass</u>:

(a) <u>Categoricity in one large enough λ</u>:

We have here concentrated on going up in cardinality, (assuming that in ω successive cardinals there are not too many models without even assuming the existence of models of cardinality $\geq \lambda^{+3}$!). We use weak instances of GCH $(2^\lambda < 2^{\lambda^+})$ and prove a generalization of [Sh 87a], [Sh 87b]. But originally, and it still seems a priori more reasonable, probably even more central case should be to start assuming categoricity in some high enough cardinal. There are several approximations in Makkai-Shelah [MaSh 285], Kolman-Shelah [KlSh 362], [Sh 472] using so called "large cardinals".

> (Compact cardinals in the first, measurable cardinal in the second and third).

(b) <u>Main Gap</u>:

If we assume that for some "large enough" λ, we do not have "many very complicated models", we expect to be able to show the class is "managable", hence has a structure theory. But the proofs described above, do not do that job. Not only do we usually start with categoricity assumptions, in our main line here we learn whatever we learn only on the $\lambda^{+\omega}$-brimmed models. However, just on the class of models, i.e., on the original $\mathfrak{K}$, we know little. This is not surprising as, e.g. for elementary classes with countable vocabulary, the solution of Łos conjecture predates the main gap considerably.

(c) <u>Superstability</u>:

Having claimed that the superstability is a central dividing lines, it is unsatisfactory to arrive at it here from categoricity assumptions only.

That is, the detailed building of apparatus parallel to superstability is here applied to cases in which we start assuming suitable categoricity assumption, prove there are relevant good λ-frames and continue. (But if $\psi \in \mathbb{L}_{\omega_1,\omega}$ or $\mathfrak{K}$ is an abstract elementary class which is $\mathrm{PC}_{\aleph_0}$ and $2^{\aleph_0} + \dot{I}(\aleph_1, \mathfrak{K}) < 2^{\aleph_1}$ this is not so: by

II§3 there is a good $\aleph_0$-frame $\mathfrak{s}$ whose $\aleph_1$-saturated models belongs to Mod_ψ, but $\mathfrak{s}$ is not necessarily uni-dimensional (which is the "internal" form of categoricity)). Probably the main weakness of beautiful λ-frames as a candidate to being the true superstable is the lack of non-structure results which are not "local". Presently, the results are about "failure of categoricity", see III§12 where for beautiful $\mathfrak{s}$, which is not uni-dimensional, we prove non-categoricity in $K_\mu^{\mathfrak{s}}$ for every $\mu >$. So natural candidate version of solvability, see [Sh 842].

(d) $\underline{\aleph_1\text{-compact structures}}$:

We may like to relax the definition of abstract elementary class to investigate classes of structures satisfying some kind of countable compactness, i.e., any reasonable countable set of demands has a solution. This will include "$\aleph_1$-saturated models" of an elementary class (even with countable vocabulary) also complete metric spaces but those are closer to elementary classes.

What we lose is closure under unions of ω-chains. For elementary classes this corresponds to $\aleph_1$-saturated models (more generally, $\mathrm{LST}(\mathfrak{K})^+$-saturated) and we have stable instead of superstable (the class of complete metric spaces is closer to elementary classes). We have considerable knowledge on the stable case but much less than on superstable ones. In particular, even for elementary classes with countable vocabulary the main gap for stable $\aleph_1$ models is not known.

(e) $\underline{\text{Some unaesthetic points in Theorem 2.17}}$

One of them is that from [Sh 576] we get (in II§3) a good λ^+-frame and not a good λ-frame. Second, we use here (in III) for simplicity in the non-structure results an extra set theoretic assumption, though a very weak one.

> Namely, the weak diamond ideal on λ^+ is not λ^{++}-saturated. The negation of this statement, if consistent, has high consistency strength. In fact, my attempts to derive good λ-frames from [Sh 576] or dealing with weaker versions had delayed Chapter II considerable.

(f) $\underline{\text{Lack of Counter-examples}}$:

By Hart-Shelah [HaSh 323], Shelah-Villaveces [ShVi 648] there are some examples for the categoricity spectrum being non-trivial. Still in many theorems on dividing lines it is not proved that they are real, i.e., that there are examples.

(g) $\underline{\text{Natural Examples}}$:

This bothers me even less than clause (f) but for many investigators the major drawback is lack of "natural examples", i.e., finding classes which are already important where the theory developed on the structure side throw light on the special case. (E.g., for simple theories, pseudo finite fields; for $\aleph_0$-stable theories, differentially closed fields of characteristic zero; for countable stable theories, differentially

closed fields of characteristic $p > 0$ (and even separably closed fields of charactertistic $p > 0$)).

But see Zilber works, especially Ravello paper.

(B) <u>The full half and half baked</u>:

Some works throw some light on some of the points from (A), in particular Chapter IV, [Sh:E46], [Sh 838]. Concerning (a), in Chapter IV we assume an abstract elementary class $\mathfrak{K}$ is categorical in large enough μ and we investigate $\mathfrak{K}_\lambda$ for $\lambda < \mu$ which are carefully chosen, specifically we assume

$(*)_\lambda$ (a) $\operatorname{cf}(\lambda) = \aleph_0$ which means $\lambda = \Sigma\{\lambda_n : n < \omega\}$ for some $\lambda_n < \lambda$

(b) $\lambda = \beth_\lambda$ which means that for every $\kappa < \lambda$ not only $2^\kappa < \lambda$ but $\beth_\kappa < \lambda$ where $\beth_\alpha$ is defined inductively by iterating exponentiation, i.e., defining inductively $\beth_\alpha = \aleph_0 + \Sigma\{2^{\beth_\beta} : \beta < \alpha\}$

<u>or</u> even

$(**)_\lambda$ $(a) + (b) + \lambda$ is the limit of cardinals λ' satisfying $(*)_\lambda$.

Are such cardinals large? Not in the set theoretic sense (i.e., provably in ZFC there are such cardinals), they are in some sense analog to the tower function in finite combinatorics. Ignoring "few" exceptional μ, a result of Chapter IV is the existence of a superlimit model in $\mathfrak{K}_\lambda$; moreover the main theorem IV.? of Chapter IV says that there is a good λ-frame $\mathfrak{s}$ with $\mathfrak{K}_\mathfrak{s} \subseteq \mathfrak{K}$; the proof uses infinitary logics. Also if the categoricity spectrum contains arbitrarily large cardinals then for some closed unbounded class $\mathbf{C}$ of cardinals, $[\lambda \in \mathbf{C} \wedge \operatorname{cf}(\lambda) = \aleph_0 \Rightarrow \mathfrak{K}$ categorical in $\lambda]$. It seems reasonable that this can be combined with Chapter III, but there are difficulties.

Having IV.? may still leave us wondering whether we have more tangible argument that we have advance. So we go back to earlier investigations of such general contexts. Now Makkai-Shelah [MaSh 285] deal with $T \subseteq \mathbb{L}_{\kappa,\omega}$ categorical in some μ big enough than $\kappa + |T|$ and develop enough theory to prove that the categoricity spectrum in an end-segment of the cardinals starting not too far, i.e. below $\beth_{(2^{\kappa+|T|})^+}$ <u>but</u>, with two extra assumptions.

First, κ is a strongly compact cardinal. This is natural as our problem is that $\mathbb{L}_{\kappa,\omega}$ lack many of the good properties of first order logic, and for strongly compact cardinals, some form of compactness is regained (even for $T \subseteq \mathbb{L}_{\kappa,\kappa}$). Still this is undesirable.

Second, we should assume that μ is a successor cardinal, this exhibit that the theory we build is not good enough. Now Kolman-Shelah [KlSh 362] + [Sh 472] partially rectify the first problem: κ is required just to be a measurable cardinal (instead of strongly compact), still measurable is not a small cardinal. Moreover, there is an extra, quite heavy price - we deal with the categoricity spectrum just

below μ and say nothing on it above so the categoricity spectrum is proved to be an interval instead of an end-segment. A parallel work [Sh 394] replace measurability by the assumption that our $\mathfrak{K}$ is an abstract elementary class with amalgamation; a major point there is trying to deal with the theory problem of locality of types (and see Baldwin [Bal0x]). Note that in both works we get amalgamation of $\mathfrak{K}$ below μ.

We address both cases together, assuming only that our abstract elementary class $\mathfrak{K}$ has the amalgamation property <u>below</u> μ. We try to eliminate those two model theoretic drawbacks: starting from a successor cardinal, and looking only below it, in IV.?, using Chapter III. For this we prove that suitable cases of failure of non-structure imply cases of $(< \mu, \kappa)$-locality[10] for saturated models (which means if $p \in \mathscr{S}_{\mathfrak{K}}(M), M \in \mathfrak{K}_{<\mu}$ is saturated then $\langle p \upharpoonright N : N \leq_{\mathfrak{K}} M, \|N\| = \kappa\rangle$ determine p). We also show that every $M \in K_N$ is quite saturated, using a generalization of the stability spectrum for linear orders from IV§6.

Finally, we conclude (also for abstract elementary class) $\mathfrak{K}$ with amalgamations assuming enough cases of $2^{\lambda} < 2^{\lambda^+}$ we can characterize the categoricity spectrum (eliminating earlier restriction to successor cardinals). This is done showing Chapter III applies, so we need the existence of enough λ, such that $\langle 2^{\lambda^{+n}} : n < \omega\rangle$ is strictly increasing.

So we have eliminated the two thorny model theoretic problems and we eliminated the use of large cardinals but we use this weak form of GCH, we intend to deal with it in [Sh 842].

Considering clause (b) from (A), the main gap, it seems far ahead. A more basic short-coming is that in III§12 we get "$\mathfrak{s}^{+\omega}$ is $\lambda_{\mathfrak{s}}^{+\omega}$-beautiful" and "for beautiful μ-frame $\mathfrak{t}$ we can prove the main gap" but this is just for, essentially, the class of $\lambda_{\mathfrak{s}}^{+\omega}$-saturated models.

Concerning (A)(c), superstability, [Sh 842] suggests "$\mathfrak{K}$ is (λ, κ)-solvable" as the true generalization of superstable (remembering superstability is schizophrenic in our context); this is weaker than categoricity and we use this assumption in Chapter IV; it is O.K. to use it always but we delay this to [Sh 842]. Essentially it means:

$\boxdot$ for some vocabulary $\tau_1 \supseteq \tau_{\mathfrak{K}}$ of cardinality κ and $\psi \in \mathbb{L}_{\kappa^+,\omega}(\tau_1), \psi$ has a model of cardinality $\geq \beth_{(2^{\kappa})^+}$ and $([M \models \psi \wedge \|M\| = \lambda \Rightarrow M \upharpoonright \tau$ is superlimit in $\mathfrak{K}]$.

A major justification for the parallelism with superstability is that for elementary classes this is equivalent to superstability.

But in [Sh 842], III§12 needs to be reworked hopefully toward the needed continuation.

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[10]called "tame" by many

We can look at results from [Sh:c] which were not regained in beautiful λ-frames. Well, of course, we are far from the main gap for the original $\mathfrak{K}$ ([Sh:c, XIII]) and there are results which are obviously more strongly connected to elementary classes, particularly ultraproducts. This leaves us with parts of type theory: semi-regular types, weight, **P**-simple[11] types, "hereditarily orthogonal to **P**" (the last two were defined and investigated in [Sh:a, V,§0 + Def4.4-Ex4.15], [Sh:c, V,§0,pg.226,Def4.4-Ex4.15,pg.277-284]). The more general case of (strictly) stable classes was started in [Sh:c, V,§5] and [Sh 429] and much advanced in Hernandes [He92].

Note that "a type q is p-simple (or **P**-simple)" and "q is hereditarily orthogonal to p (or **P**)" are essentially the[12] "internal" and "foreign" in Hrushovski's profound works.

Some years ago [Sh 839] started to deal with this to some extent. No problem to define weight, but for having "simple" types we need to be somewhat more liberal in the definition of abstract elementary class - allow function symbols of infinite arity (= number of places) while preserving the uniqueness of direct limit. In the right form which includes the case of $\aleph_1$-saturated models of a stable theory, we generalize what was known (for elementary classes); see more in 4.9 and before.

Lastly, considering (A)(e), to a large extent this is resolved as a product of redoing and extending the non-structure theory of [Sh 576] in [Sh 838].

In view of I§5 it is natural to weaken the stability demand in the definition of a good λ-frame to $M \in K_{\mathfrak{s}} \Rightarrow |\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)| \leq \lambda^+_{\mathfrak{s}}$ and this is called a semi-good λ-frame. (The present way is to choose a countable close enough set of types and redefine $(K, \leq_{\mathfrak{K}})$ so we restrict the class of models. Semi-good frames are introduced and investigated by Jarden-Shelah [JrSh 875]. Concerning clause (A)(f), Baldwin-

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[11]The motivation is for suitable **P** (e.g. a single regular type) that on the one hand $\mathrm{stp}(a, A) \pm$ **P** $\Rightarrow$ $\mathrm{stp}(a/E, A)$ is **P**-simple for some equivalence relation definable over A and on the other hand if $\mathrm{stp}(a_i, A)$ is **P**-simple for $i < \alpha$ then $\Sigma\{w(a_i, A) \cup \{a_j : j < i\}) : i < \alpha\}$ does not depend on the order in which we list the a_i's. Note that **P** here is $\mathscr{P}$ there.

[12]Note, "foreign to **P**" and "hereditarily orthogonal to **P** are equivalent. Now (**P** $= \{p\}$ for ease)

   (a) $q(x)$ is $p(x)$-simple when for some set A, in $\mathfrak{C}$ we have $q(\mathfrak{C}) \subseteq \mathrm{acl}(A \cup \bigcup p_i(\mathfrak{C}))$

   (b) $q(x)$ is $p(x)$-internal when for some set A, in $\mathfrak{C}$ we have $q(\mathfrak{C}) \subseteq \mathrm{dcl}(A \cup p(\mathfrak{C}))$.

Note

   (α) internal implies simple

   (β) if we aim at computing weights it is better to stress acl as it covers more

   (γ) but the difference is minor and

   (δ) in existence it is better to stress dcl, also it is useful that $\{F \restriction (p(\mathfrak{C}) \cup q(\mathfrak{C}) : F$ an automorphism of $\mathfrak{C}$ over $p(\mathfrak{C}) \cup \mathrm{Dom}(p)\}$ is trivial when $q(x)$ is p-internal but not so for p-simple (though form a pro-finite group).

Shelah [BlSh 862] expands our knowledge of examples considerably. Concerning clause (A)(g) see Zilber [Zi0xa], [Zi0xb].

In [Sh:F709] may try to axiomatize the end of I§5 and connect it to good $\aleph_0$-frames, [Sh:E54] will say more on Chapter II. In [Sh 838] we also deal with the positive theory of almost good frame and weak versions of $K_{\mathfrak{s}}^{3,\mathrm{uq}}$. Also [Sh:F735] will consider redoing Chapter III under weaker assumptions and getting more and [Sh:F782] will continue Chapter IV, e.g. how the good λ-frame from IV§4 fit Chapter III. Also [Sh:F888] will try to continue [Sh:E56], and [Sh:F841] to continue [Sh 838].

(C) <u>The white part of the map</u>:

So we would really like to know

<u>4.1 Problem</u>: What can be the categoricity spectrum Cat-Spec$_{\mathfrak{K}}$ = $\{\lambda : \mathfrak{K}$ is categorical$\}$ for an abstract elementary class ?

This seems too hard at present and involves independence results. Note also that easily (by known results, see [Ke70] or see ([Sh:c, VII,§5]) for any $\alpha < \omega_1$ for some abstract elementary class $\mathfrak{K}$ (with LST$(\mathfrak{K}) = \aleph_0$) we have: $\lambda \in$ Cat-Spec$_{\mathfrak{K}} \Leftrightarrow \lambda >$ $\beth_\alpha$ (just let $\psi = \psi_1 \vee \psi_2 \in \mathbb{L}_{\omega_1,\omega}(\tau), \psi_1$ has a model of cardinality λ iff $\lambda \leq \beth_\alpha$ and ψ_2 says that all predicates and function symbols are trivial).

Considering the history it seemed to me that the main question on our agenda should be

<u>4.2 Conjecture</u>: If $\mathfrak{K}$ is an abstract elementary class then either every large enough λ belongs to Cat-Spec$_{\mathfrak{K}}$ <u>or</u> every large enough λ does not belong to Cat-Spec$_{\mathfrak{K}}$ (provably in ZFC).

After (or you may say if) this is resolved positively we should consider

4.3 Conjecture. 1) If $\mathfrak{K}$ is an a.e.c. with LST$(\mathfrak{K}) = \chi$ then

    (a) Cat-Spec$_{\mathfrak{K}}$ includes or is disjoint to $[\beth_\omega(\chi), \infty)$
        or even better
    $(a)^+$ similarly for $[\lambda_\omega, \infty)$ where $\lambda_0 = \chi, \lambda_{n+1} = \min\{\lambda : 2^\lambda > 2^{\lambda_n}\}, \lambda_\omega = \Sigma\{\lambda_n : n < \omega\}$

probably more realistic are

    (b) similarly for $[\beth_{(2^\chi)^+}, \infty)$, or at least
    (c) similarly $(\beth_{1,1}(\chi), \infty)$ or at least $(\beth_{1,\omega^\omega}(\chi), \infty)$, see IV§0.

This will be parallel in some sense to the celebrated investigations of the countable models for (first order) countable T categorical in $\aleph_1$.

Further questions are: (recall $\boxdot$ above)

<u>4.4 Question</u>: What can be $\{(\lambda, \kappa) : \mathfrak{K}_\lambda$ is (λ, κ)-solvable, $\lambda \gg \kappa \gg \mathrm{LST}(\mathfrak{K})\}$?

Question 4.4 seems to us to be more profound than the categoricity spectrum as solvability is a form of superstability. We conjecture that the situation is as in 4.3(c); note that solvability seems close to categoricity and we have a start on it (Chapter IV, [Sh 842]).

Still more easily defined (but a posteri too early for us) is:

<u>4.5 Question</u>: 1) What can be $\{\lambda : \mathfrak{K}_\lambda$ has a superlimit model$\}$?

2) Similarly for locally superlimit (see IV.?).

3) For suitable Φ what can be $\{\lambda$: if I is a linear order of cardinality λ then $\mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ is pseudo superlimit$\}$? see IV.?(3).

   We conjecture it will be a variant of 4.3 but will be harder and even:

4.6 Conjecture. If $\lambda > \beth_{1,1}(\mathrm{LST}_{\mathfrak{K}})$ (or $\lambda > \beth_{1,\omega}(\mathrm{LST}_{\mathfrak{K}})$, <u>then</u> $\mathfrak{K}$ has a superlimit model in λ iff $\mathfrak{K}$ is $(\lambda, \mathrm{LST}_{\mathfrak{K}})$-solvable.

We now return to (D, λ)-homogeneous models. Of course, for special D's we may be interested in some special classes of models, but not necessarily the elementary sub-models of $\mathfrak{C}$. Of course, parallely to the first order case, the main gap for them is an important problem (e.g. the class of existentially closed models of a universal first order theory is a natural and important case). But the most natural main case seems to me the "$\mathfrak{C}$ is (D, κ)-sequence homogeneous" context:

<u>4.7 Problem</u>: Prove the main gap for the class of (D, κ)-sequence-homogeneous $M \prec \mathfrak{C}$; considering what we know, we can assume $\kappa \geq \kappa(D)$, see [Sh 3] (and §1(B)) and concentrate on $\kappa \geq \aleph_1$ and we would like to prove that

   (a) either the number of such models of cardinality $\aleph_\alpha = \aleph_\alpha^{<\kappa(D)} + \lambda(D)$ is small, i.e., $\leq \beth_{\gamma(D)}(|\alpha|)$ for[13] some $\gamma(D)$ not depending on α <u>or</u> the number is $2^{\aleph_\alpha}$ (where $\lambda(D)$ is the first "stability cardinal" of D).

   (b) $\gamma(D)$ does not depend on κ.

A parallel of "the main gap for the class of $\aleph_\varepsilon$-saturated models of a first order T" in this context is dealt with in Hyttinen-Shelah [HySh 676], and a parallel to the "main gap for the class of model of a totally transcendental first order T" in

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[13]of course, $\beth_{\gamma(D)}(|\alpha|)$ may be $\geq 2^{\aleph_\alpha}$ in which case this says little; this consistently occurs for every $\alpha \geq \omega$. But if G.C.H. holds, and if we ask on $\dot{I}\dot{E}(\lambda, -)$ for the class we get clear cut results

Grossberg-Lessman [GrLe0x], and surely there is more to be said in those cases <u>but</u> in the problem above, even the case $\kappa = \aleph_1, \mathfrak{C}$ saturated is not covered.

We hope eventually to find a stability theory for the "countably compact abstract elementary class" strong enough to prove as a special case the main gap for the $\aleph_1$-saturated models of elementary classes (i.e., clause (d) of (A)) as said above maybe [Sh 839] help.

The reader may wonder: if not known for elementary classes why you expect more from a general frame? Of course, we do not know, but:

<u>4.8 Thesis</u>: The better closure properties of the abstract frames should help us, being able to, e.g., make induction on frames.

Hence
<u>4.9 Thesis</u>: Some problems on elementary classes are better dealt with in some non-elementary contexts (close to abstract elementary class), as if we would like during the proof to consider some derived other classes, those contexts give you more freedom. In particular this may apply to

(*a*) main gap for $|T|^+$-saturated models (the parallel of (D, λ)-sequence-homogeneous above in 4.7 and (d) of (A) and discussion on it in (B))

(*b*) the main gap for the class of models of T for an <u>uncountable</u> first order T.

Note that [Sh 300], Chapter II has tried to materialize this, but that program is not finished.

<u>4.10 Problem</u>: Similar questions for the number of pairwise non-elementarily embeddable (D, λ)-sequence homogeneous models.

In the case of the class of models (not the class of $\aleph_1$-saturated models) for countable first order theories, those two problems were solved together.

There are many other interesting questions in this context. An important one, of a different character is:
<u>4.11 Problem</u>: 1) [Hanf number for sequence homogeneous]
Given a cardinal κ, what is the first λ such that: if T is a complete first order theory, $D \subseteq D(T) = \{\operatorname{tp}(\bar{a}, \emptyset, M) : M$ a model of $T, \bar{a} \in {}^{\omega>}M\}$ and there is a (D, λ)-sequence-homogeneous model, <u>then</u> for every $\mu > \lambda$ there is a (D, μ)-sequence homogeneous model.
2) Similarly for $\{\kappa$: in $\mathfrak{K}$ we have amalgamation for models of cardinality $< \kappa$ (and $\kappa \geq \operatorname{LST}(\mathfrak{K}) > \aleph_0\})$.
3) Similarly for $(\mathbb{D}, \lambda)$-model homogeneous models (see V.B§3).

Toward this we may define semi-beautiful classses as in III§12 (or [Sh 87a], [Sh 87b]) replacing the stable $\mathscr{P}^-(n)$-systems by an abstract notion, omitting uniqueness and the definability of types and retaining existence. Semi-excellent classes seem like an effective version of having amalgamation, so it certainly implies it; such properties may serve as what we actually have to prove to solve the problem 4.11 above. We may have to use more complicated frames: say classes $\mathfrak{K}_n$ so that $M \in \mathfrak{K}_n$ is actually a $\mathscr{P}^-(n)$-system of models from $\mathfrak{K}$. (See more in [Sh 842]).

Recall that a class $\mathfrak{K}$ of structures with fixed vocabulary τ is called <u>universal</u> if it is closed under isomorphisms, and $M \in \mathfrak{K}$ <u>if and only if</u> every finitely generated submodel of M belongs to $\mathfrak{K}$. So not every elementary class is a universal class, but many universal classes are not first order (e.g., locally finite groups). This investigation leads (see [Sh 300], Chapter II) to classes with an axiomatized notion of non-forking and much of [Sh:c] was generalized, sometimes changing the context (a case of Thesis 4.9), but, e.g., still:

<u>4.12 Problem</u>: Prove the main gap for the universal context.

<u>4.13 Question</u>: Can we in [Sh 576], i.e. [Sh:E46] weaken the "categorical in λ^+" to "has a superlimit model in λ^+"?

See on this hopefully [Sh:F888].

<u>4.14 Question</u>: Do we use a parallel of III§12 with existential side for serious effect? (See more in [Sh 842]).

§5 Basic knowledge

(A) What knowledge needed and dependency of the chapters

The chapters were written separately, hence for better or for worse there are some repetitions, hopefully helping the reader if he likes to read only parts of this book.

Chapter III depends on Chapter II and [Sh:E46] depends somewhat, e.g. on II§1, but in other cases there are no real dependency.

In fact, reading Chapter II, Chapter III requires little knowledge of model theory, they are quite self-contained, in particular you do not need to know Chapter I, Chapter II; this apply also to Chapter II and to [Sh:E46]. Of course, if a claim proves that the axioms of good λ-frames are satisfied by the class of models of a sentence ψ in a logic you have not heard about, it will be a little loss for you to ignore the claim (this occurs in II§3). Still much of the material is motivated by parallelism to what we know in elementary (= first order) contexts. Let me stress that neither do we see any merit in not using large model theoretic background nor was its elimination an a priori aim, but there is no reason to hide this fact from a potential reader who may feel otherwise.

Also the set theoretic knowledge required in Chapter II, Chapter III is small; still we use cardinals and ordinals of course, induction on ordinals, cofinality of an ordinal, so regular cardinals, see here below for what you need. A priori it seemed that somewhat more is needed in the proof of the non-structure theorems, i.e., showing a class with a so-called "non-structure property" has many, complicated models so cannot have a structure theory. But we circumvent this by quoting [Sh 838], or you can say delaying the proof. That is, we carry the construction enough to give a reasonable argument. So the reader can just agree to believe; similarly in Chapter II and in [Sh:E46].

In [Sh 838] itself, we rely somewhat on basics of II§1, and in the applications ([Sh 838, §4]) we somewhat depend on the relevant knowledge and for [Sh 838, §5-§8] we assume the basics of II§2. Also [Sh 838, §9,§10,§11] are set theoretic, mainly use results on the weak diamond which we quote.

The situation is different in Chapter I. Still you can read §1, §2, §3 of it ignoring some claims but in §4,§5 the infinitary logics $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ and its relatives and basic theorems on them are important.

For Chapter IV you need basic knowledge of infinitary logics and Ehenfeucht-Mostowski models, and in IV§4 (the main theorem) we use the definition of good λ-frame from II§2.

(B) Some basic definitions and notation

We first deal with model theory and then with set theory.

5.1 Definition. 1) A vocabulary τ is a set of function symbols (denoted by G, H, F) and relation symbols, (denoted by P, Q, R) (= predicates), to each such symbol a number of places (= arity) is assigned (by τ) denoted by $\mathrm{arity}_\tau(F)$, $\mathrm{arity}_\tau(P)$, respectively.
2) M is a τ-model or a τ-structure for a vocabulary τ means that M consists of:

- (a) its universe, $|M|$, a non-empty set
- (b) P^M, the interpretation of a predicate $P \in \tau$ and P^M is an $\mathrm{arity}_\tau(P)$-place relation on $|M|$
- (c) F^M, the interpretation of a function symbol $F \in \tau$ and F^M is an $\mathrm{arity}_\tau(F)$-place function from $|M|$ to $|M|$ in the case of arity $0, F^M$ is an individual constant.

3) We agree τ is determined by M and denote it by τ_M. If $\tau_1 \subseteq \tau_2, M_2$ a τ_2-model, then $M_1 = M_2 \restriction \tau_1$, the reduct is naturally defined.
4) The cardinality of $M, \|M\|$, is the cardinality, number of elements of the universe $|M|$ of M. We may write $a \in M$ instead of $a \in |M|$ and $\langle a_i : i < \alpha \rangle \in M$ instead $i < \alpha \Rightarrow a_i \in M$, i.e., $\bar{a} \in {}^\alpha|M|$.
5) Let $M \subseteq N$ mean that

$$\tau_M = \tau_N, |M| \subseteq |N|, P^M = P^N \restriction |M|, F^M = F^N \restriction |M|$$

for every predicate $P \in \tau_M$ and for every function symbol $F \in \tau_M$.
6) If N is a τ-model and A is a non-empty subset of $|M|$ closed under F^N for each function symbol $F \in \tau$, <u>then</u> $N \restriction A$ is the unique $M \subseteq N$ with universe A.

5.2 Definition. 1) K denotes a class of τ-models closed under isomorphisms, for some vocabulary $\tau = \tau_K$.
2) $\mathfrak{K}$ denotes a pair $(K, \leq_\mathfrak{K})$; K as above (with $\tau_\mathfrak{K} := \tau_K$) and $\leq_\mathfrak{K}$ is a two-place relation on K closed under isomorphisms such that $M \leq_\mathfrak{K} N \Rightarrow M \subseteq N$.
3) f is a $\leq_\mathfrak{K}$-embedding of M into N <u>when</u> for some $N' \leq_\mathfrak{K} N, f$ is an isomorphism from M onto N'.
4) K is categorical in λ <u>if</u> K has one and only one model up to isomorphism of cardinality λ. If $\mathfrak{K} = (K, \leq_\mathfrak{K})$ we may say "$\mathfrak{K}$ is categorical in λ".

5.3 Definition. 1) For a class K (or $\mathfrak{K}$) of τ_K-models

- (a) $K_\lambda = \{M \in K : \|M\| = \lambda\}$
- (b) $\mathfrak{K}_\lambda = (K_\lambda, \leq_\mathfrak{K} \restriction K_\lambda)$

(c) $\dot{I}(\lambda, K) = \dot{I}(\lambda, \mathfrak{K}) = |\{M/\cong: M \in K_\lambda\}|$ so K (or $\mathfrak{K}$) is categorical in λ iff $\dot{I}(\lambda, K) = 1$

(d) $\dot{I}\dot{E}(\lambda, \mathfrak{K}) = \sup\{\mu$: there is a sequence $\langle M_\alpha : \alpha < \mu \rangle$ of members of K_λ such that M_α is not $\leq_{\mathfrak{K}}$-embeddable into M_β for any distinct $\alpha, \beta < \mu\}$. But writing $\dot{I}\dot{E}(\lambda, \mathfrak{K}) \geq \mu$ we mean the supremum is obtained if not said otherwise.

(e) $M \in \mathfrak{K}$ is $(\leq_{\mathfrak{K}}, \lambda)$-universal if every $N \in \mathfrak{K}_\lambda$ can be $\leq_{\mathfrak{K}}$-embedded into it. If $\lambda = \|M\|$ we may write $\leq_{\mathfrak{K}}$-universal. If $\mathfrak{K}$ is clear from the context we may write λ-universal or universal (for $\mathfrak{K}$).

We end the model-theory part by defining logics (this is not needed for Chapter II, Chapter III, [Sh:E46] and Chapter II except some parts of Chapter V.A).

5.4 Definition. A logic $\mathscr{L}$ consists of:

(a) function $\mathscr{L}(-)$ (actually a definition) giving for every vocabulary τ a set of so-called formulas $\varphi(\bar{x}), \bar{x}$ a sequence of free variables with no repetitions

(b) $\models_{\mathscr{L}}$, satisfaction relation, i.e., for every vocabulary τ and $\varphi(\bar{x}) \in \mathscr{L}(\tau)$ and τ-model M and $\bar{a} \in {}^{\ell g(\bar{x})}M$ we have "$M \models_{\mathscr{L}} \varphi[\bar{a}]$" or in words "$M$ satisfies $\varphi[\bar{a}]$"; holds or fails.

As for set theory

5.5 Definition. 1) A power = number of elements of a set, is identified with the first ordinal of this power, that is a cardinal. Such ordinals are called cardinals, $\aleph_\alpha$ is the α-th infinite ordinal.
2) Cardinals are denoted by $\lambda, \mu, \kappa, \chi, \theta, \partial$ (infinite if not said otherwise).

5.6 Definition. 0) Ordinals are denoted by $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \xi, i, j$, but, if not said otherwise δ denotes a limit ordinal.
1) An ordinal α is a limit ordinal if $\alpha > 0$ and $(\forall \beta < \alpha)[\beta + 1 < \alpha]$.
2) For an ordinal α, $\mathrm{cf}(\alpha)$, the cofinality of α, is $\min\{\mathrm{otp}(u) : u \subseteq \alpha$ is unbounded$\}$; it is a regular cardinal (see below), we can define the cofinality for linear orders and again get a regular cardinal.
3) A cardinal λ is regular if $\mathrm{cf}(\lambda) = \lambda$, otherwise it is called singular.
4) If $\lambda = \aleph_\alpha$ then $\lambda^+ = \aleph_{\alpha+1}$, the successor of λ, so $\lambda^{++} = \aleph_{\alpha+2}, \lambda^{+\varepsilon} = \aleph_{\alpha+\varepsilon}$.

Recall:

5.7 Claim. *1) If λ is a regular cardinal, $|\mathcal{U}_t| < \lambda$ for $t \in I$ and $|I| < \lambda$ <u>then</u> $\cup\{\mathcal{U}_t : t \in I\}$ has cardinality $< \lambda$.*
2) λ^+ is regular for any $\lambda \geq \aleph_0$ but $\lambda^{+\delta}$ is singular if δ is a limit ordinal $< \lambda$ (or just $< \lambda^{+\delta}$), and, obviously, $\aleph_0$ is regular but e.g. $\aleph_\omega$ is singular, in fact $\aleph_\delta > \delta \Rightarrow \aleph_\delta$ is singular, but the inverse is false.

Sometimes we use (not essential)

5.8 Definition/Claim. 1) $\mathcal{H}(\lambda)$ is the set of x such that there is a set Y of cardinality $< \lambda$ which is transitive (i.e. $(\forall y)(y \in Y \Rightarrow y \subseteq Y)$ and x belongs to λ.
2) Every x belongs to $\mathcal{H}(\lambda)$ for some x.
So for some purpose we can look at $\mathcal{H}(\lambda)$ instead of the universe of all sets.

§6 Index of symbols[14]

a member of a model

A set of elements of model

$\mathfrak{A}$ a "complicated" model

b member of a model

B set of members of models

$\mathfrak{B}$ a "complicated" model

c member of model (also individual constant)

$\mathbf{c}$ colouring, mainly [Sh 838]

C set or elements of models <u>or</u> a club

$\mathscr{C}$ club of $[A]^{<\lambda}$,

$\mathfrak{C}$ a complicated model, <u>or</u> a monster

d member of model

$\mathbf{d}$ expanded I-system, III§12; $\mathfrak{u}$-free rectangle or triangle in [Sh 838]

D diagram; set of $(<\omega)$-types in the first order sense realized in a model, Chapter I, Chapter V.B

$\mathbf{D}$ a function whose values are diagrams, Chapter I, Chapter V.B

$\mathbb{D}$ diagram for model homogeneity, Chapter I, so set of isomorphism types of models, also Chapter V.B

$\mathfrak{D}$ a set of $\mathbb{D}$'s, Chapter V.B

$\mathscr{D}$ filter

$\mathscr{D}_\lambda$ club filter on the regular cardinal $\lambda > \aleph_0$

e element of a model <u>or</u> a club

$\mathbf{e}$ expanded I-system (used in continuations), III§12; $\mathfrak{u}$-free rectangle or triangle in [Sh 838]

E a club

$\mathbb{E}$ filter

$\mathscr{E}$ an equivalence relations, (e.g. $\mathscr{E}_M, \mathscr{E}_M^{\mathrm{at}}$ in II.? for definition of type and $\mathscr{E}_{\mathfrak{K},\chi}, \mathscr{E}_{\mathfrak{K},\chi}^{\mathrm{mat}}$ in V.B§3)

f function (e.g., isormorphism, embedding usually)

$\mathbf{f}$ function ([Sh 838] in $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K_{\mathfrak{u}}^{3,\mathrm{qt}}$, also in II§5, $(\bar{M}, \bar{\mathbf{f}}), (\bar{M}, \bar{\mathbf{J}}, \mathbf{f}))$

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[14]some will be used only in subsequent works; in particular concerning forcing

F function symbol

$\mathbb{F}$ amalgamation choice function ([Sh 838] also see [Sh 576, §3])

$\mathbf{F}$ function (complicated, mainly it witnesses a model being limit, I§3)

g function

$\mathfrak{g}$ witness for almost every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f})$ see [Sh 838, 1a.43-1a.51]

G function symbol

$\supset$ game

h function

$\mathfrak{h}$ witnesses for almost every $(\bar{M}, \bar{\mathbf{J}}, \mathbf{f}) \in K_\mathfrak{u}^{\mathrm{qt}}$, see [Sh 838, c.4A-c.4D] or [Sh 838, 1a.43-1a.51]

H function symbol

$\mathscr{H}$ in $\mathscr{H}(\lambda)$, rare here see 5.8

i ordinal/natural number

I linear order, partial order or index set

$\dot{I}$ $\dot{I}(\lambda, K)$, numbers on non-isomorphic models; $\dot{I}\dot{E}(\lambda, K)$ (see Chapter I), also $\dot{I}(K)$, see [Sh 838]

$\mathbf{I}$ set of sequences or elements from a model, in particular:

     $\mathbf{I}_{M,N} = \{c \in N : \mathbf{tp}_\mathfrak{s}(c, M, N) \in \mathscr{S}_\mathfrak{s}^{\mathrm{bs}}(M)\}$, see Chapter II, Chapter III

$\check{I}[\lambda]$ a specific normal ideal, see I§0, marginal here

$\mathbb{I}$ ideal

$\mathscr{I}$ predense set in a forcing $\mathbb{P}$, very rare here

j ordinal/natural number

J linear order, index set, Chapter I

$\mathbf{J}$ set of sequences or elements from a model

$\mathbb{J}$ ideal

$\mathscr{J}$ predense set in a forcing $\mathbb{P}$, very rare here

k natural number

K class of model of a fix vocabulary $\tau_\mathfrak{K}$, K_λ is $\{M \in K : \|M\| = \lambda\}$

$\mathfrak{K}$ is $(K, \leq_\mathfrak{K})$, usually abstract elementary class

$K_\mathfrak{s}^{3,x}$ for $x = \{\mathrm{bs,uq,pr,qr,vq,bu}\}$, appropriate set of triples (M, N, a) or $(M, N, \mathbf{I})$, see Chapter II, Chapter III

$K_\lambda^{3,\mathrm{na}}$ for triples (M, N, a), see [Sh:E46]

$K_\mathfrak{u}^{3,x}$ set of triples $(M, N, \mathbf{J}) \in \mathrm{FR}_\mathfrak{u}^\ell$, see [Sh 838]

ℓ natural number

L language (set of formulas, e.g., $\mathscr{L}(\tau)$ but also subsets of $\mathscr{L}(\tau)$ which normally are closed under subformulas and first order operations), used in Chapter I.

LST   Löwenheim-Skolem-Tarski numbers, mainly $\mathrm{LST}(\mathfrak{K}) = \mathrm{LST}_{\mathfrak{K}}$

$\mathscr{L}$ logic, i.e., a function such that $\mathscr{L}(\tau)$ is a language for vocabulary τ (but also a language mainly $\mathscr{L}$ a fragment of $\mathbb{L}_{\lambda^+,\omega}$, i.e., a subset closed under subformulas and the finitary operations)

$\prec_{\mathscr{L}}$ is used for $M \prec_{\mathscr{L}} N$ iff $M \subseteq N$ and for every $\varphi(\bar{x}) \in \mathscr{L}(\tau_M)$ and $\bar{a} \in {}^{\ell g(\bar{x})}M$ we have $M \models \varphi[\bar{a}] \Leftrightarrow N \models \varphi[\bar{a}]$

$\mathbb{L}$ first order logic and $\mathbb{L}_{\lambda,\kappa}, \mathbb{L}^{\ell}_{\lambda,\kappa}$, see Chapter I so $\varphi(\bar{x}) \in \mathbb{L}_{\lambda,\kappa}$ has $< \kappa$ free variables

L   the constructible universe

m natural number

m an I-system in III§12

M model

M   complicated object, see [Sh:E46, §3,§4]

n natural number

n an I-system in III§12, for continuation and in Chapter V.F

N model

$\mathbb{N}$ the natural numbers

p type

p   member of $\mathbb{P}$, a forcing condition, very rare here

P predicate

$\mathscr{P}$ power set, family of sets,

P   family of types, Chapter III

$\mathbb{P}$ forcing notion, very rare here

q type

q   forcing condition, very rare here

Q predicate

Q   a quantifier written $(\mathbf{Q}x)\varphi$, see Chapter I, if clear from the context means $\mathbf{Q}^{\mathrm{car}}_{\geq \aleph_1}$

$\mathbf{Q}^{\mathrm{car}}_{\geq \kappa}$ the quantifier there are $\geq \kappa$ many

$\mathbb{Q}$ the rationals

r type

r   forcing condition, very rare here

R predicate

$\mathbb{R}$ reals

s member of I, J

$\mathfrak{s}$ frame

S set of ordinals, stationary set many times

$\mathscr{S}$ $\mathscr{S}_{\mathfrak{K}}(M)$ is a set of types in the sense of orbits, $\mathscr{S}_{\mathfrak{s}}^{\mathrm{bs}}(M)$ the basic types (there are some alternatives to bs)

S $\mathbf{S}_L^{\alpha}(A, M)$ set of complete (L, α)-types over M, so a set of formulas, used when we are dealing with a logic $\mathscr{L}$, may use $\mathbf{S}_{\mathscr{L}}^{\alpha}(A, M)$

$\mathbb{S}$ $\mathbb{S}(M)$ is a set of pseudo types, are neither set of formulas nor orbits, but formal non-forking extension (for continuations, see [Sh 842])

t member of I, J

tp   type as set of formulas

tp   type as an orbit, an equivalence class under mapping

t   type function

$\mathfrak{t}$ frame

T first order theory, usually complete

$\mathscr{T}$ a tree

u a set

$\mathfrak{u}$ a nice construction framework, in [Sh 838]

unif   in $\mu_{\mathrm{unif}}(\lambda, 2^{<\lambda})$, see I.? or [Sh 838, 0z.6](6)

U a set

$\mathscr{U}$ a set

v a set

V a set

V   universe of set theory

w a set

W a set (usually of ordinals)

$\mathscr{W}$ a class of triples $(N, \bar{M}, \bar{\mathbf{J}})$; see III§7

wd   in $\mu_{\mathrm{wd}}(\lambda)$ see I§0, [Sh 838, §0]

WDmId_{λ} the weak diamond ideal, see I.?

x variable (or element)

x   complicated object, in [Sh 838] such that is a sequence

$$\langle(\bar{M}^\alpha, \bar{\mathbf{J}}^\alpha, \mathbf{f}^\alpha) : \alpha < \alpha(*)\rangle$$

X set

y variable

$\mathbf{y}$ like $\mathbf{x}$

Y set

$\mathscr{Y}$ a high order variable (see I§3)

z variable

Z set

$\mathbb{Z}$ the integers

Greek Letters:

α ordinal

β ordinal

γ ordinal

Γ various things; in [Sh:E46] a set of models or types

δ ordinal, limit if not clear otherwise

∂ cardinal

Δ set of formulas (may be used for symmetric difference)

ϵ ordinal

ε ordinal

ζ ordinal

η sequence, usually of ordinals

θ cardinal, infinite if not clear otherwise

ϑ a formula, very rare

Θ set of cardinals/class of cardinals

ι ordinal (sometimes a natural number)

κ cardinal, infinite if not clear otherwise

λ cardinal, infinite if not clear otherwise

$\lambda(\mathfrak{K})$ is the L.S.T.-number of an abstract elementary class $(\geq |\tau_\mathfrak{K}|$ for simplicity), rare

Λ set of formulas, used in Chapter IV, Chapter I

μ cardinal, infinite if not said otherwise

ν sequence, usually of ordinals

σ a term (in a vocabulary τ)

Σ sum

π permutation

Π product

ρ sequence, usually of ordinals

ϱ sequence, usually of ordinals

τ vocabulary (so $\mathscr{L}(\tau), \mathbb{L}(\tau), \mathbb{L}_{\lambda,\mu}(\tau)$ are languages)

ξ ordinal

Ξ a complicated object

Υ ordinal and other objects

χ cardinal, infinite if not said otherwise

φ formula

Φ blueprint for EM-models

ψ formula

Ψ blueprint for EM-models

ω the first infinite ordinal

Ω a complicated object

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